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Editorial for the Special Issue 100 Years of Chronogeometrodynamics: The Status of the Einstein's Theory of Gravitation in Its Centennial Year

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The present Editorial introduces the Special Issue dedicated by the journal *Universe* to the General Theory of Relativity, the beautiful theory of gravitation of Einstein, a century after its birth. It reviews some of its key features in a historical perspective, and, in welcoming distinguished researchers from all over the world to contribute it, some of the main topics at the forefront of the current research are outlined.

general relativity and gravitation; classical general relativity; gravitational waves; quantum gravity; cosmology; experimental studies of gravity

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1. Introduction

This year marks the centenary of the publication of the seminal papers [1–3] in which Albert Einstein laid down the foundations of his theory of gravitation, one of the grandest achievements of the human thought which is the best description currently at our disposal of such a fundamental interaction shaping the fabric of the natural world. It is usually termed “General Theory of Relativity” (GTR, from *Allgemeine Relativitätstheorie*), often abbreviated as “General Relativity” (GR). It replaced the Newtonian concept of “gravitational force” with the notion of “deformation of the chronogeometric structure of spacetime” [4] due to all forms of energy weighing it; as such, it can be defined as a chronogeometrodynamical theory of gravitation [5].

GTR is connected, in a well specific sense, to another creature of Einstein himself, with Lorentz [6] and Poincaré [7,8] as notable predecessors, published in 1905 [9]: the so-called Special (or Restricted)

Theory of Relativity (STR). The latter is a physical theory whose cornerstone is the requirement of covariance of the differential equations expressing the laws of physics (originally only mechanics and electromagnetism) under Lorentz transformations of the spacetime coordinates connecting different inertial reference frames, in each of which they must retain the same mathematical form. More precisely, if

$$A(x, y, z, t), B(x, y, z, t), C(x, y, z, t), \dots \quad (1)$$

represent the state variables of a given theory depending on spacetime coordinates x, y, z, t and are mutually connected by some mathematical relations

$$f(A, B, C, \dots) = 0 \quad (2)$$

representing the theory's fundamental equations, the latter ones can always be mathematically written in a covariant form under a generic transformation from the old coordinates to the new ones as

$$f'(A'(x', y', z', t'), B'(x', y', z', t'), C'(x', y', z', t'), \dots) = 0 \quad (3)$$

In general, the new functional relations f' connecting the transformed state variables A', B', C', \dots are different from the ones of f . If, as for the Lorentz transformations, it turns out

$$f' = f \quad (4)$$

which does not necessarily implies that also the state variables A, B, C, \dots remain unchanged, then it is said that the equations of the theory retain the same form. It is just the case of the Maxwell equations, in which the electric and magnetic fields \mathbf{E}, \mathbf{B} transform in a given way under a Lorentz transformation in order to keep the form of the equations connecting them identical, which, instead, is not retained under Galilean transformations [10]. In the limiting case of the Galilean transformations applied to the Newtonian mechanics, it turns out that the theory's equations are even invariant in the sense that also the state variables remain unchanged, *i.e.*, it is

$$\mathbf{F}' - m\mathbf{a}' = 0 \quad (5)$$

with

$$\begin{aligned} \mathbf{F}' &= \mathbf{F} \\ \mathbf{a}' &= \mathbf{a} \end{aligned} \quad (6)$$

As such, strictly speaking, the key message of STR is actually far from being: “everything is relative”, as it might be seemingly suggested by its rather unfortunate name which, proposed for the first time by Planck [11] (*Relativtheorie*) and Bucherer [12] (*Relativitätstheorie*), became soon overwhelmingly popular (see also [13]). Suffice it to say that, in informal correspondence, Einstein himself would have preferred that its creature was named as *Invariantentheorie* (Theory of invariants) [14], as also explicitly proposed-unsuccessfully-by Klein [15]. Note that, here, the adjective “invariant” is used, in a looser sense, to indicate the identity of the mathematical functional form connecting the transformed state variables.

Notably, if the term “relativity” is, instead, meant as the identity of all physical processes in reference frames in reciprocal translational uniform motion connected by Lorentz transformations, then, as remarked by Fock [16], a name such as “Theory of Relativity” can, to some extent, be justified. In *this specific sense*, relativity geometrically corresponds to the maximal uniformity of the pseudo-Euclidean spacetime of Poincaré and Minkowski in which it is formulated. Indeed, given a N -dimensional manifold, which can have constant curvature, or, if with zero curvature, can be Euclidean or pseudo-Euclidean, the group of transformations which leave identical the expression for the squared distance between two nearby points can contain at most $(1/2)N(N + 1)$ parameters. If there is a group involving all the $(1/2)N(N + 1)$ parameters, then the manifold is said to have maximal uniformity. The most general Lorentz transformations, which leave unchanged the coefficients of the expression of the 4-dimensional distance between two nearby spacetime events, involve just 10 parameters. Now, in the pseudo-Riemannian spacetime of GTR the situation is different because, in general, it is not uniform at all in the geometric sense previously discussed. Following Fock [16], it can be effectively illustrated by a simple example whose conclusion remains valid also for the geometry of the 4-dimensional spacetime manifold. Let us think about the surface of a sphere, which is a 2-dimensional manifold of a very particular form. It is maximally uniform since it can be transformed into itself by means of rotations by any angle about an arbitrary axis passing through the centre, so that the associated group of transformations has just three parameters. As a result, on a surface of a sphere there are neither preferred points nor preferred directions. A more general non-spherical surface of revolution has only partial uniformity since it can be transformed into itself by rotation about an axis which is now fixed, so that the rotation angle is the only arbitrary parameter left. There are privileged points and lines: the poles through which the axis passes, meridians, and latitude circles. Finally, if we consider a surface of general form, there will be no transformations taking it into itself, and it will possess no uniformity whatsoever. Thus, it should be clear that the generality of the form of the surface is a concept antagonistic to the concept of uniformity. Returning now to the concept of relativity *in the aforementioned specified sense*, it is related to uniformity in all those cases in which the spacetime metric can be considered fixed. This occurs not only in the Minkowskian spacetime, but also in the Einsteinian one, provided only that the physical processes one considers have no practical influence on the metric. Otherwise, it turns out that relativity can, to a certain extent, still be retained only if the non-uniformity generated by heavy masses may be treated as a local perturbation in infinite Minkowskian spacetime. To this aim, let us think about a laboratory on the Earth’s surface [16]. If it was turned upside down, relativity would be lost since the physical processes in it would be altered. But, if the upset down laboratory was also parallel transported to the antipodes, relativity would be restored since the course of all the processes would be the same as at the beginning. In this example, a certain degree of relativity was preserved, even in a non-uniform spacetime, because the transformed gravitational field g' in the new coordinate system $\{x'\}$ has the same form as the old field g in the old coordinates $\{x\}$, *i.e.*,

$$\{x\} \mapsto \{x'\} \quad (7)$$

$$g(x) \mapsto g'(x') = g(x) \quad (8)$$

Such considerations should have clarified that relativity, in the previously specified sense, either does not exist at all in a non-uniform spacetime like the Einsteinian one, or else it does exist, but does *not* go

beyond the relativity of the Minkowskian spacetime. In this sense, the gravitational theory of Einstein cannot be a *generalization of his theory of space and time of 1905*, and its notion of relativity along with its related concept of maximal uniformity was *not* among the concepts subjected to *generalization*. Since the greatest possible uniformity is expressed by Lorentz transformations, there *cannot* be a *more general* principle of relativity than that discussed in the theory of 1905. All the more, there cannot be a general principle of relativity having physical meaning which would hold with respect to arbitrary frames of references. As such, both the denominations of “General Relativity” and “General Theory of Relativity” are confusing and lead to misunderstandings. Furthermore, such adjectives reflect also an incorrect understanding of the theory itself since they were adopted referring to the covariance of the equations with respect to arbitrary transformations of coordinates accompanied by the transformations of the coefficients of the distance between two events in the 4-dimensional spacetime. But it turned out that such kind of covariance has actually nothing to do with the uniformity or non-uniformity of spacetime [16,17]. Covariance of equations *per se* is just a merely mathematical property which in no way is expression of any kind of physical law. Suffice it to think about the Newtonian mechanics and the physically equivalent Lagrange equations of second kind which are covariant with respect to arbitrary transformations of the coordinates. Certainly, nobody would state that Newtonian mechanics contains in itself “general” relativity. A principle of relativity-Galilean or Einsteinian-implies a covariance of equations, but the converse is not true: covariance of differential equations is possible also when no principle of relativity is satisfied. Incidentally, also the the adjective “Special” attached to the theory of 1905 seems improper in that it purports to indicate that it is a special case of “General” Relativity.

In the following, for the sake of readability, we will adhere to the time-honored conventions by using STR and GTR (or GR) for the Einsteinian theory of space and time of 1905 and for his gravitational theory of 1915, respectively.

Of course, the previous somewhat “philosophical” considerations are, by no means, intended to undermine the credibility and the reliability of the majestic theory of gravitation by Einstein, whose concordance with experiments and observations has been growing more and more over the latest decades [18].

Below, some key features of GTR, to which the present Special Issue is meritoriously and timely dedicated, are resumed in a historical perspective [19–21], without any pretence of completeness. It is hoped that the distinguished researchers who will kindly want to contribute it will provide the community of interested readers with the latest developments at the forefront of the research in this fascinating and never stagnant field.

In the following, Greek letters $\mu, \nu, \varrho \dots$ denote 4-dimensional spacetime indexes running over 0, 1, 2, 3, while Latin ones i, j, k, \dots , taking the values 1, 2, 3, are for the 3-dimensional space.

2. The Incompatibility of the Newtonian Theory of Gravitation with STR

In the framework of the Newtonian theory of universal gravitation [22], the venerable force-law yielding the acceleration \mathbf{a} imparted on a test particle by a mass distribution of density ρ could be

formally reformulated in the language of the differential equations governing a field-type state variable Φ , known as potential, through the Poisson equation [23]

$$\nabla^2 \Phi = 4\pi G \rho \quad (9)$$

where G is the Newtonian constant of gravitation, so that

$$\mathbf{a} = -\nabla \Phi \quad (10)$$

Nonetheless, although useful from a mathematical point of view, such a field was just a non-dynamical entity, deprived of any physical autonomous meaning: it was just a different, mathematical way of telling the same thing as the force law actually did [20]. It is so because, retrospectively, in the light of STR, it was as if, in the Newtonian picture, the gravitational interaction among bodies would take place *de facto* instantaneously, irrespectively of the actual distance separating them, or as if gravity would be some sort of occult, intrinsic property of matter itself. Remarkably, such a conception was opposed by Newton himself who, in the fourth letter to R. Bentley in 1692, explicitly wrote [24]: “[...] Tis inconceivable that inanimate brute matter should (without the mediation of something else which is not material) operate upon & affect other matter without mutual contact; as it must if gravitation in the sense of Epicurus be essential & inherent in it. And this is one reason why I desired you would not ascribe {innate} gravity to me. That gravity should be innate inherent & {essential} to matter so that one body may act upon another at a distance through a vacuum without the mediation of any thing else by & through which their action or force {may} be conveyed from one to another is to me so great an absurdity that I believe no man who has in philosophical matters any competent faculty of thinking can ever fall into it. Gravity must be caused by an agent {acting} constantly according to certain laws, but whether this agent be material or immaterial is a question I have left to the consideration of my readers.”. In the previous quotation, the text in curly brackets { . . . } is unclear in the manuscript, but the editor of the original document is highly confident of the reading.

In the second half of the nineteenth century, with the advent of the Maxwellian field theory of electromagnetism [25] scientists had at disposal a mathematically coherent and empirically well tested model of a physical interaction among truly dynamical fields which propagate as waves even *in vacuo* at the finite speed of light c transferring energy, momentum and angular momentum from a point in space to another. Now, STR is based on two postulates: The Principle of Relativity, extended by Einstein to all physical interactions, and another principle that states that the speed of light is independent of the velocity of the source. In this form, it retains its validity also in GTR. The latter is an immediate consequence of the law of propagation of an electromagnetic wave front which is straightforwardly obtained from the Maxwell equations obeying, by construction, the Principle of Relativity itself since they turned out to be covariant under Lorentz transformations. It necessarily follows [16] that there exists a maximum speed for the propagation of any kind of physical action. This is numerically equal just to the speed of light *in vacuo*. If there was no single limiting velocity but instead different agents, e.g., light and gravitation, propagated *in vacuo* with different speeds, then the Principle of Relativity would necessarily be violated as regards at least one of the the agents. Indeed, it would be possible to choose an inertial frame traveling just at the speed of the slower agent in which the differential equations governing its course would take a particular form with respect to that assumed in all the other frames, thus predicting

spurious, unphysical phenomena. It is reminiscent of the famous first *gedankenexperiment* made by Einstein about STR around 1895-1896 described by himself as follows [26]: “[...] Wenn man einer Lichtwelle mit Lichtgeschwindigkeit nachläuft, so würde man ein zeitunabhängiges Wellenfeld vor sich haben. So etwas scheint es aber doch nicht zu geben!” [“If one goes after a light wave with light velocity, then one would have a time-independent wavefield in front of him. However, something like that does not seem to exist!”] Indeed, the Maxwell equations *in vacuo*, in their known form, do not predict stationary solutions. That posed severe challenges to the Newtonian gravitational theory [8], which necessarily would have had to abandon its strict force-law aspect in favor of a genuine field-type framework making the Poisson equation covariant under Lorentz transformations [19,27].

Furthermore, as pointed out by Einstein himself [28], Newtonian universal gravitation did not fit into the framework of the maximally uniform spacetime of SRT for the deepest reason that [16], while in SRT the inertial mass m_i of a material system had turned out to be dependent on its total energy, in the Newtonian picture the gravitational mass m_g , did not. At high speeds, when the change in the inertia of a body becomes notable, this would imply a breakdown of the law of free fall, whose validity was actually well tested, although only at non-relativistic regimes (see Section 3).

Finally, it can be remarked also that the required Lorentz covariance would have imposed, in principle, also the existence of a new, magnetic-type component of the gravitational field so to yield some sort of gravitational inductive phenomena and travelling waves propagating at the finite speed of light *in vacuo*. Unfortunately, at the dawn of the twentieth century, there were neither experimental nor observational evidence of such postulated manifestations of a somehow relativistic theory of gravitation.

3. The Equivalence Principle and Its Consequences

3.1. The Equality of the Inertial and Gravitational Masses Raised to the Status of a Fundamental Principle of Nature

Luckily, at that time, Einstein was pressed also by another need: The quest for a coherent framework to consistently write down the laws of physics in arbitrary frames of references moving according to more complicated kinematical laws than the simple uniform translation. In 1907 [29], Einstein realized that the bridge across such two apparently distinct aspects could have been represented by the equality of the inertial and gravitational masses, known at that time to a 5×10^{-8} accuracy level thanks to the Eötvös experiment [30].

That was an empirical fact well known since the times of Galilei thanks to the (likely) fictional [31–33] tales of his evocative free fall experiments [34] allegedly performed from the leaning tower of Pisa around 1590. Newton himself was aware of the results by Galilei, and made his own experiments with pendulums of various materials obtaining an equality of inertial and gravitating masses to a 10^{-3} level of relative accuracy. Indeed, in the Proposition VI, Theorem VI, Book III of his *Principia* [22] Newton wrote [35]: “It has been, now for a long time, observed by others, that all sorts of heavy bodies [...] descend to the Earth from equal heights in equal times; and that equality of times we may distinguish to a great accuracy, by the help of pendulums. I tried experiments with gold, silver, lead, glass, sand, common salt, wood, water, and wheat. I provided two wooden boxes, round and equal: I filled the one

with wood, and suspended an equal weight of gold (as exactly as I could) in the centre of oscillation of the other. The boxes, hanging by equal threads of 11 feet, made a couple of pendulums perfectly equal in weight and figure, and equally receiving the resistance of the air. And, placing the one by the other, I observed them to play together forwards and backwards, for a long time, with equal vibrations. And therefore the quantity of matter in the gold (by Cors. I and VI, Prop. XXIV, Book II) was to the quantity of matter in the wood as the action of the motive force (or *vis motrix*) upon all the gold to the action of the same upon all the wood; that is, as the weight of the one to the weight of the other: and the like happened in the other bodies. By these experiments, in bodies of the same weight, I could manifestly have discovered a difference of matter less than the thousandth part of the whole, had any such been.” Interestingly, in the Proposition VI, Theorem VI, Book III of his *Principia* [22], Newton looked also the known motions of the natural satellites of Jupiter to make-from a phenomenological point of view -a further convincing case for the equality of the inertial and gravitational masses. Indeed, if the ratios of the gravitational to the inertial mass of Jupiter and of its satellites were different, the orbits of the Jovian moons about their parent planet would be unstable because of an imperfect balancing of the centrifugal acceleration and the Jupiter centripetal attraction caused by a residual, uncanceled force due to the Sun’s attractions on either Jupiter and its moons themselves. Indeed, Newton wrote [36]: “[...] that the weights of Jupiter and of his satellites towards the Sun are proportional to the several quantities of their matter, appears from the exceedingly regular motions of the satellites (by Cor. III, Prop. LXV, Book I). For if some of those bodies were more strongly attracted to the Sun in proportion to their quantity of matter than others, the motions of the satellites would be disturbed by that inequality of attraction (by Cor. II, Prop. LXV, Book I). If, at equal distances from the Sun, any satellite, in proportion to the quantity of its matter, did gravitate towards the Sun with a force greater than Jupiter in proportion to his, according to any given proportion, suppose of d to e ; then the distance between the centres of the Sun and of the satellite’s orbit would be always greater than the distance between the centres of the Sun and of Jupiter, nearly as the square root of that proportion: as by some computations I have found. [...]” In principle, the Newtonian gravitational theory would have not lost its formal consistency even if experiments-all conducted at low speeds with respect to c -would have returned a different verdict about m_i/m_g . Nonetheless, one cannot help but notice as the very same name chosen by Newton for the universally attractive force regulating the courses of the heavens, *i.e.*, gravitation, may point, somehow, towards a not so accidental nature of the equality of inertial and gravitating masses. Indeed, it comes from the Latin word *gravis* (‘heavy’) with several Indoeuropean cognates [37], all with approximately the same meaning related to the weight of common objects on the Earth’s surface: Sanskrit, *guruḥ* (‘heavy, weighty, venerable’); Greek, βάρος (‘weight’) and βαρύς (‘heavy in weight’); Gothic, *kaurus* (‘heavy’); Lettish, *gruts* (‘heavy’). It is tempting to speculate that, perhaps, Newton had some sort of awareness of the fundamental nature of that otherwise merely accidental fact. It seems not far from the position by Chandrasekhar who wrote [38]: “There can be no doubt that Newton held the *accurate proportionality of the weight ‘to the masses of matter which they contain’* as inviolable”.

Whatever the case, Einstein promoted it to a truly *fundamental* cornerstone on which he erected his beautiful theoretical building: the Equivalence Principle (EP). Indeed, the postulated *exact* equality of the inertial and gravitational mass implies that, in a given constant and uniform gravitational field, all bodies move with the same acceleration in exactly the same way as they do in an uniformly accelerated reference

frame removed from any external gravitational influence. In this sense, an uniformly accelerated frame in absence of gravity is equivalent to an inertial frame in which a constant and uniform gravitational field is present. It is important to stress that the need of making the universality of the free fall, upon which the EP relies, compatible with the dictates of the SRT was not at all a trivial matter [21] (cfr. Section 1), and the merit of keeping the law of free fall as a *fundamental* principle of a viable relativistic theory of gravitation which could not reduce to a mere extension of the Newtonian theory to the SRT must be fully ascribed to Einstein. To better grasp the difficulties posed by such a delicate conceptual operation, let us think about an inertial reference frame K in which two stones, differing by shape and composition, move under the action of a uniform gravitational field starting from the same height but with different initial velocities; for the sake of simplicity, let us assume that, while one of the two stones is thrown horizontally with an initial velocity with respect to K , the other one falls vertically starting at rest [21]. Due to the universality of free fall, both the stones reach the ground simultaneously. Let us, now, consider an inertial frame K' moving uniformly at a speed equal to the horizontal component of the velocity of the projectile; in this frame, the kinematics of the two objects gets interchanged: the projectile has no horizontal velocity so that now it falls vertically, while the stone at rest acquires an horizontal velocity making it move parabolically in the opposite direction with respect to K' . According to the universality of the free fall, also in this case they should come to the rest at the same time. But this is in disagreement with the relativity of the simultaneity of the SRT. Moreover, another source of potential tension between the universality of the free fall and the SRT is as follows [21]. According to the latter one, a change in the energy of a body corresponds to a change also in its inertial mass, which acts as a “brake”. On the other hand, since the inertial mass is equivalent to the gravitational mass, which, instead, plays the role of “accelerator”, the correct relativistic theory of gravitation necessarily implies that also the gravitational mass should depend in an exactly known way from the total energy of the body. Actually, other scientists like, e.g., Abraham [27] and Mie [39] were willing to discard the Galileo’s law of universality of free fall to obtain a relativistic theory of gravitation.

The heuristic significance of the original form of the EP unfolded in the findings by Einstein that identical clocks ticks at different rates if placed at different points in a gravitational potential, a feature which was measured in a laboratory on the Earth’s surface in 1960 [40] by means of the Mössbauer effect which has recently received a general relativistic interpretation [41], and the gravitational redshift of the spectral lines emitted at the Sun’s surface with respect to those on the Earth, which was measured only in the sixties of the last century [42] following the 1925 measurement with the spectral lines in the companion of Sirius [43]. Furthermore, it turned out that the speed of light in a gravitational field is variable, and thus light rays are deflected, as if not only an inertial mass but also a gravitational mass would correspond to any form of energy. Einstein [28] was also able to calculate the deflection of the apparent position of background stars due to the Sun’s gravitational potential, although the value he found at that time was only half of the correct one later predicted with the final form of his GTR [44] and measured in 1919 [45,46] (see Section 4). In 1912, he [47,48] explored the possibility of gravitational lensing deriving the basic features of the lensing effect, which will be measured for the first time not until 1979 [49]. It must be noted [19,21] that this theory of the constant and uniform gravitational field went already *beyond* STR. Indeed, because of the dependence of the speed of light and the clock rates on the gravitational potential, STR definition of simultaneity and the Lorentz transformation themselves

lost their significance (cfr. Section 1). *In this specific sense*, it can be said that STR can hold only in absence of a gravitational field.

The existence of non-uniformly accelerated reference frames like, e.g., those rotating with a time-dependent angular velocity $\boldsymbol{\Omega}(t)$, naturally posed the quest for a further generalization of the EP able to account for spatially and temporally varying gravitational fields as well. The extension of the EP to arbitrarily accelerating frames necessarily implies, in principle, the existence of further, non-uniform, non-static (either stationary and non-stationary) and velocity-dependent gravitational effects, as guessed by Einstein [50–52]. They were later fully calculated by Einstein himself [53] and others [54–60] with the final form of the GTR (see Section 4 and [61–63] for critical analyses of the seminal works), which could not be encompassed by the gravito-static Newtonian framework. Indeed, it must be recalled that the inertial acceleration experienced by a body (slowly) moving with velocity \mathbf{v}' with respect to a rotating frame K' is

$$\mathbf{a}'_{\Omega} = 2\boldsymbol{\Omega} \times \mathbf{v}' + \dot{\boldsymbol{\Omega}} \times \mathbf{r}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}') \quad (11)$$

At least to a certain extent, such new gravitational effects, some of which have been measured only a few years ago [64–67], might be considered as reminiscent of the Machian relational conceptions of mechanics [68–71].

Such a generalization of the EP to arbitrary gravitational fields lead Einstein to reformulate it as follows: in any infinitesimal spacetime region (*i.e.*, sufficiently small to neglect either spatial and temporal variations of gravity throughout it), it is always possible to find a suitable non-rotating coordinate system K_0 in which any effect of gravity on either test particle motions and any other physical phenomena is absent. Such a local coordinate system can ideally be realized by a sufficiently small box moving in the gravitational field freely of any external force of non-gravitational nature. Obviously, it appeared natural to assume the validity of STR in K_0 in such a way that all the reference frames connected to it by a Lorentz transformation are physically equivalent. *In this specific sense*, it could be said that the Lorentz covariance of all physical laws is still valid in the infinitely small.

At this point, still relying upon the EP, it remained to construct a theory valid also for arbitrarily varying gravitational fields by writing down the differential equations connecting the gravitational potential, assumed as state variable, with the matter-energy sources and requiring their covariance with respect to a fully general group of transformations of the spacetime coordinates.

3.2. Predictions of the Equivalence Principle

A step forward was done in 1914 when, in collaboration with Grossmann, Einstein [72], on the basis of the Riemannian theory of curved manifolds, was able to introduce the ten coefficients $g_{\mu\nu}$ of the symmetric metric tensor \mathbf{g} by writing down the square of the spacetime line element $(ds)^2$ between two infinitely near events in arbitrary curvilinear coordinates x^μ as

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (12)$$

As a consequence, the equations of motion of a test particle, the energy-momentum theorem and the equations of the electromagnetism *in vacuo* were simultaneously written in their generally covariant ultimate form. In particular, from the right-hand-side of the geodesic equation of motion of a test particle

$$\frac{d^2 x^\alpha}{ds^2} = -\Gamma_{\beta\varrho}^\alpha \frac{dx^\beta}{ds} \frac{dx^\varrho}{ds} \quad (13)$$

where the Christoffel symbols

$$\Gamma_{\beta\varrho}^\alpha \doteq \frac{1}{2} g^{\alpha\sigma} \left(\frac{\partial g_{\sigma\beta}}{\partial x^\varrho} + \frac{\partial g_{\sigma\varrho}}{\partial x^\beta} - \frac{\partial g_{\beta\varrho}}{\partial x^\sigma} \right) \quad (14)$$

are constructed with the first derivatives of $g_{\mu\nu}$, it was possible to straightforwardly identify the components of \mathbf{g} as the correct state variables playing the role of the Newtonian scalar potential Φ . Indeed, to a first-order level of approximation characterized by neglecting terms quadratic in v/c and the squares of the deviations of the $g_{\mu\nu}$ from their STR values

$$\begin{aligned} \eta_{00} &= +1 \\ \eta_{ij} &= -\delta_{ij} \end{aligned} \quad (15)$$

the geodesic equations of motion for the spatial coordinates become

$$\frac{d^2 x^i}{dt^2} = -c^2 \Gamma_{00}^i \quad (16)$$

Furthermore, if the gravitational field is assumed static or quasi-static and the time derivatives can be neglected, the previous equations reduce to

$$\frac{d^2 x^i}{dt^2} = \frac{c^2}{2} \frac{\partial g_{00}}{\partial x^i} \quad (17)$$

By posing

$$\Phi \doteq -\frac{1}{2} c^2 (g_{00} - 1) \quad (18)$$

so that

$$g_{00} = 1 - \frac{2\Phi}{c^2} \quad (19)$$

the Newtonian acceleration is obtained. The additive constant up to which the potential is defined is fixed in such a way that Φ vanishes when g_{00} assumes its STR value η_{00} . It is worthwhile remarking that, to the level of approximation adopted, only g_{00} enters the equations of motion, although the deviations of the other metric coefficients from their STR values may be of the same order of magnitude. It is this circumstance that allows to describe, to a first order approximation, the gravitational field by means of a single scalar potential.

In analogy with the geodesic equations of motion for a test particle, also those for the propagation of electromagnetic waves followed. Indeed, the worldlines of light rays are, thus, geodesics curves of null length

$$\begin{aligned} (ds)^2 &= 0 \\ \frac{d^2 x^\alpha}{d\lambda^2} &= -\Gamma_{\beta\varrho}^\alpha \frac{dx^\beta}{d\lambda} \frac{dx^\varrho}{d\lambda} \end{aligned} \quad (20)$$

where λ is some affine parameter.

The components of the metric tensor g are not assigned independently of the matter-energy distributions, being determined by field equations.

A further consequence of EP and the fact that, to the lowest order of approximation, g_{00} is proportional to the Newtonian potential Φ is that, in general, it is possible to predict the influence of the gravitational field on clocks even without knowing all the coefficients $g_{\mu\nu}$; such an influence is actually determined by g_{00} through

$$d\tau = \sqrt{g_{00}}dt \quad (21)$$

where τ is the reading of a clock at rest. Instead, it is possible to predict the behaviour of measuring rods only knowing all the other coefficients g_{0i} , g_{ik} . Indeed, it turns out that the square of the distance dl between two nearby points in the 3-dimensional space is given by [73]

$$(dl)^2 = \left(-g_{jh} + \frac{g_{0j}g_{0h}}{g_{00}} \right) dx^j dx^h \quad (22)$$

Thus, the field g determines not only the gravitational field, but also the behaviour of clocks and measuring rods, *i.e.*, the chronogeometry of the 4-dimensional spacetime which contains the geometry of the ordinary 3-dimensional space as a particular case. Such a fusion of two fields until then completely separated—metric and gravitation—should be regarded as a major result of GTR, allowing, in principle, to determine the gravitational field just from local measurements of distances and time intervals.

4. The Field Equations for the Metric Tensor and Their Physical Consequences

4.1. The Field Equations

The differential equations for the g tensor itself followed in 1915 [1–3].

The tortuous path [21] which lead to them can be sketchily summarized as follows [19]. According to the EP, the gravitational mass of a body is exactly equal to its inertial mass and, as such, it is proportional to the total energy content of the body. The same must, then, hold also in a given gravitational field for the force experienced by a body which is proportional to its (passive) gravitational mass. It is, thus, natural to assume that, conversely, only the energy possessed by a material system does matter, through its (active) gravitational mass, as far as its gravitational field is concerned. Nonetheless, in STR the energy density is not characterized by a scalar quantity, being, instead, the 00 component of the so-called stress-energy tensor T . It follows that also momentum and stresses intervene on the same footing as energy itself. These considerations lead to the assumption that no other material state variables than the components $T_{\mu\nu}$ of T must enter the gravitational field equations. Moreover, in analogy with the Poisson equation, T must be proportional to a differential expression G of the second order containing only the state variables of the gravitational field, *i.e.*, the components of the metric tensor g ; because of the required general covariance, G must be a tensor as well. The most general expression for it turned out to be

$$G_{\mu\nu} = c_1 R_{\mu\nu} + c_2 g_{\mu\nu} R + c_3 g_{\mu\nu} \quad (23)$$

where R is the contracted curvature tensor whose components are

$$R_{\mu\nu} = \frac{\partial \Gamma_{\mu\alpha}^{\alpha}}{\partial x^{\nu}} - \frac{\partial \Gamma_{\mu\nu}^{\alpha}}{\partial x^{\alpha}} + \Gamma_{\mu\alpha}^{\beta} \Gamma_{\nu\beta}^{\alpha} - \Gamma_{\mu\nu}^{\alpha} \Gamma_{\alpha\beta}^{\beta} \quad (24)$$

and R is its invariant trace. The coefficients c_1 , c_2 , c_3 were determined by imposing that the stress-energy tensor satisfies the energy-momentum conservation theorem. By neglecting the third term in \mathbf{G} , which usually plays a negligible role in the effects which will be discussed in this Section (see Section 5 for phenomena in which it may become relevant), the Einstein field equations became [1,2]

$$\mathbf{G} = -\kappa \mathbf{T} \quad (25)$$

with

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad (26)$$

and κ is a constant which is determined by comparison with the Newtonian Poisson equation. By contraction, one gets

$$R = \kappa T \quad (27)$$

where T is the trace of \mathbf{T} , so that

$$R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \right) \quad (28)$$

This is the generally covariant form of the gravitational field equations to which, after many attempts, Einstein came in 1915 [3].

The same field equations were obtained elegantly by Hilbert through a variational principle [74]. On the reciprocal influences between Einstein and Hilbert in the process of obtaining the GTR field equations and an alleged priority dispute about their publication, see [75].

It should be noted [19] that GTR, *per se*, yields neither the magnitude nor the sign (attraction or repulsion of the gravitational interaction) of κ which are, instead, retrieved from the observations. For weak and quasi-static fields generated by pressureless, extremely slowly moving matter of density ρ , the right-hand-side of the field equation for the 00 component becomes

$$-\frac{1}{2}\kappa c^2 \rho \quad (29)$$

indeed, the only non-vanishing component of the matter stress-energy tensor is

$$T_{00} = \rho c^2 \quad (30)$$

so that

$$T = -\rho c^2 \quad (31)$$

Since the time derivatives and the products of the Christoffel symbols can be neglected, the 00 component of the Ricci tensor reduces to

$$R_{00} = \frac{1}{2}\nabla^2 g_{00} = -\frac{\nabla^2 \Phi}{c^2} \quad (32)$$

Thus, it is

$$\nabla^2 \Phi = \frac{1}{2}\kappa c^4 \rho \quad (33)$$

the Poisson equation really holds. A comparison with the Newtonian equation tells that κ is positive, being equal to

$$\kappa = \frac{8\pi G}{c^4} = 2 \times 10^{-43} \text{ kg}^{-1} \text{ m}^{-1} \text{ s}^2 \quad (34)$$

the spacetime can, thus, be assimilated to an extremely rigid elastic medium.

4.2. First Predictions of the Theory and Confrontation with Observations

In the same year [44], Einstein readily employed his newborn theory to successfully explain the long-standing issue of the anomalous perihelion precession of Mercury [76]. To this aim, and also in order to derive the correct value of the deflection of a light ray grazing the Sun's limb [44] through the Fermat principle, it was necessary to know not only the coefficient g_{00} of the gravitational field of a point mass, as in the Newtonian approximation, but also the other metric coefficients g_{ij} . Since the spacetime outside a spherical body is isotropic, the off-diagonal metric coefficients g_{0i} are identically zero: otherwise, they would induce observable effects capable of distinguishing between, e.g., two opposite spatial directions (see Section 3.2). Moreover, it was also required to approximate g_{00} itself to a higher order. Einstein [44] solved that problem by successive approximations. The *exact* vacuum solution was obtained one year later by Schwarzschild [77] and, independently, Droste [78]; their results are virtually indistinguishable from those of Einstein. Relevant simplifications were introduced one year later by Weyl [79], who used cartesian coordinates instead of spherical ones, and worked on the basis of the action principle instead of recurring to the differential equations for the field g . Schwarzschild [80] extended the validity of his solution also to the interior of a material body modelled as a sphere of incompressible fluid. Having in hand this exact solution of the Einstein field equations revolutionized the successive development of GTR. Indeed, instead of dealing only with small weak-field corrections to Newtonian gravity, as Einstein had initially imagined would be the case, fully nonlinear features of the theory such as gravitational collapse and singularity formation could be studied, as it became clear decades later. About the Schwarzschild solution, the Birkhoff's Theorem [81] was proved in 1923. According to it, even without the assumption of staticity, the Schwarzschild metric is the *unique* vacuum solution endowed with spherically symmetry. As a consequence, the external field of a spherical body radially pulsating or radially imploding/exploding is not influenced at all by such modifications of its source.

The successful explanation of the anomalous perihelion precession of Mercury was a landmark for the validity of GTR since, as remarked in [82,83], it was a successful *retrodiction* of an effect which was known for decades. In particular, Weinberg wrote [83]: “It is widely supposed that the true test of a theory is in the comparison of its predictions with the results of experiment. Yet, with the benefit of hindsight, one can say today that Einstein's successful explanation in 1915 of the previously measured anomaly in Mercury's orbit was a far more solid test of general relativity than the verification of his calculation of the deflection of light by the sun in observations of the eclipse of 1919 or in later eclipses. That is, in the case of general relativity a *retrodiction*, the calculation of the already-known anomalous motion of Mercury, in fact provided a more reliable test of the theory than a true *prediction* of a new effect, the deflection of light by gravitational fields.

I think that people emphasize prediction in validating scientific theories because the classic attitude of commentators on science is not to trust the theorist. The fear is that the theorist adjusts his or her theory to fit whatever experimental facts are already known, so that for the theory to fit to these facts is not a reliable test of the theory.

But [...] no one who knows anything about how general relativity was developed by Einstein, who at all follows Einstein's logic, could possibly think that Einstein developed general relativity in order to

explain this precession. [...] Often it is a successful *prediction* that one should really distrust. In the case of a true prediction, like Einstein's prediction of the bending of light by the sun, it is true that the theorist does not know the experimental result when she develops the theory, but on the other hand the experimentalist does know about the theoretical result when he does the experiment. And that can lead, and historically has led, to as many wrong turns as overreliance on successful retrodictions. I repeat: it is not that experimentalists falsify their data. [...] But experimentalists who know the result that they are theoretically supposed to get naturally find it difficult to stop looking for observational errors when they do not get that result or to go on looking for errors when they do. It is a testimonial to the strength of character of experimentalists that they do not always get the results they expect".

The final work of Einstein on the foundations of GTR appeared in 1916 [84].

In the same year, de Sitter [85] was able to derive a further consequence of the static, spherically symmetric spacetime of the Schwarzschild solution: the precession of the orbital angular momentum of a binary system, thought as a giant gyroscope, orbiting a non-rotating, spherical body such as in the case of the Earth-Moon system in the Sun's field. Some years later, Schouten [86] and Fokker [87] independently obtained the same effect by extending it also to spin angular momenta of rotating bodies. Such an effect is mainly known as de Sitter or geodetic precession. It was measured decades later in the field of the Sun by accurately tracking the orbit of the Earth-Moon system with the Lunar Laser Ranging technique [88,89], and in the field of the Earth itself with the dedicated Gravity Probe B (GP-B) space-based experiment [64] and its spaceborne gyroscopes.

In 1964 [90], Shapiro calculated a further prediction of the static Schwarzschild spacetime: The temporal delay, which since then bears his name, experienced by travelling electromagnetic waves which graze the limb of a massive body as the Sun in a back-and-forth path to and from a terrestrial station after having been sent back by a natural or artificial body at the superior conjunction with our planet. In its first successful test performed with radar signals [91], Mercury and Venus were used as reflectors. Latest accurate results [92] relied upon the Cassini spacecraft en route to Saturn.

4.3. The General Approximate Solution by Einstein

In 1916 [93], Einstein, with a suitable approximation method, was able to derive the field generated by bodies moving with arbitrary speeds, provided that their masses are small enough. In this case, the $g_{\mu\nu}$ differ slightly from the STR values $\eta_{\mu\nu}$, so that the squares of their deviations $h_{\mu\nu}$ with respect to the latter ones can be neglected, and it is possible to keep just the linear part of the field equations. Starting from their form [1,2]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu} \quad (35)$$

working in the desired approximation, they can be cast into a linearized form in terms of the auxiliary state variables

$$\bar{h}_{\mu\nu} \doteq h_{\mu\nu} - \frac{1}{2}\delta_{\mu}^{\nu}h \quad (36)$$

where δ_{μ}^{ν} is the Kronecker delta, and h is the trace of \mathbf{h} which is a tensor only with respect to the Lorentz transformations. A further simplification can be obtained if suitable spacetime coordinates, satisfying the gauge condition

$$\frac{\partial \bar{h}_{\alpha\beta}}{\partial x^\beta} = 0 \quad (37)$$

known as Lorentz gauge (or Einstein gauge or Hilbert gauge or de Donder gauge or Fock gauge), are adopted. The resulting differential equations for the state variables $\bar{h}_{\mu\nu}$ are

$$\square \bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu} \quad (38)$$

which is the inhomogeneous wave equation; \square is the STR form of the d'Alembertian operator. The usual method of the retarded potentials allows to obtain

$$\bar{h}_{\mu\nu} = \frac{\kappa}{2\pi} \int \frac{T_{\mu\nu}(x', y', z', t - r/c)}{r} dx' dy' dz' \quad (39)$$

Among other things, it implies that the action of gravity propagates to the speed of light: a quite important results which, some years ago, was the subject of dispute [94–96] boosted by the interpretation of certain VLBI measurements of the time delay suffered at the limb of Jupiter by electromagnetic waves from distant quasars [97,98].

4.3.1. Gravitational Waves

The form of the gravitational waves in empty regions follows from the Lorentz gauge condition and the inhomogeneous wave equation by posing $T = 0$: it was studied by Einstein in [99], where he also calculated the emission and the absorption of gravitational waves. It turned out that, when oscillations or other movements take place in a material system, it emits gravitational radiation in such a way that the total power emitted along all spatial directions is determined by the third temporal derivatives of the system's moment of inertia

$$I_{ij} = \int \rho x^i x^j dx^1 dx^2 dx^3 \quad (40)$$

Instead, when a gravitational wave impinges on a material system whose size is smaller than the wave's wavelength, the total power absorbed is determined by the second temporal derivatives of its moment of inertia [99].

Gravitational waves were *indirectly* revealed for the first time [100–102] in the celebrated Hulse-Taylor binary pulsar PSR B1913+16 [103,104]. *Direct* detection (some of) their predicted effects in both terrestrial [105–110] and space-based laboratories [111–117] from a variety of different astronomical and astrophysical sources [118], relentlessly chased by at least fifty years since the first proposals by Gertsenshtein and Pustovoit [119] of using interferometers and the pioneering attempts by J. Weber [120] with its resonant bars [121], is one of the major challenges of the current research in relativistic physics [122,123].

Conversely, by assuming their existence, they could be used, in principle, to determine key parameters of several extreme astrophysical and cosmological scenarios which, otherwise, would remain inaccessible to us because of lack of electromagnetic waves from them [124] by establishing an entirely new “Gravitational Wave Astronomy” [122,125]. A recent example [126] is given by the possibility that the existence of primordial gravitational waves may affect the polarization of the electromagnetic radiation which constitutes the so-called Cosmic Microwave Background (CMB),

discovered in 1965 [127]. In this case, the polarizing effect of gravity is indirect since the field of the gravitational waves does not directly impact the polarization of CMB, affecting, instead, the anisotropy of the spatial distribution of CMB itself. Indeed, the polarization of CMB is a direct consequence of the scattering of the photons of the radiation with the electrons and positrons which formed the primordial plasma, existing in the primordial Universe at the so-called decoupling era [128]. At later epochs, when the temperature fell below 3000 K°, the radiation decoupled from matter, photons and electrons started to interact negligibly, and the polarization got “frozen” to the values reached at the instant of decoupling. Thus, mapping the current CMB’s polarization state has the potential of providing us with direct information of the primordial Universe, not contaminated by the dynamics of successive evolutionary stages. In particular, it turns out that the presence of metric fluctuations of tensorial type, *i.e.*, of gravitational waves, at the epoch in which the CMB radiation interacted with the electrons of the cosmic matter getting polarized, may have left traces in terms of polarization modes of B type [129,130]. They could be currently measurable, provided that the intensity of the cosmic background of gravitational waves is strong enough. An example of cosmic gravitational background able to produce, in principle, such an effect is represented by the relic gravitational radiation produced during the inflationary epochs. The gravitational waves produced in this way are distributed over a very wide frequency band $\Delta\omega(t)$ which is generally time-dependent. In order to characterize the intensity of such relic gravitational waves, it turns out convenient to adopt the spectral energy density

$$\varepsilon_h(\omega, t) \doteq \frac{d\varepsilon(t)}{d \ln \omega} \quad (41)$$

defined as the energy density $\varepsilon(t)$ per logarithmic interval of frequency, normalized to the critical energy density $\varepsilon_{\text{crit}}$ (see Section 5.4), *i.e.*, the dimensionless variable

$$\Omega_h(\omega, t) \doteq \frac{1}{\varepsilon_{\text{crit}}} \frac{d\varepsilon}{d \ln \omega} \quad (42)$$

The simplest inflationary models yield power-law signatures for it. In 2014 [131], the BICEP2 experiment at the South Pole seemed to have successfully revealed the existence of the B modes; the measured values seemed approximately in agreement—at least in the frequency band explored by BICEP2—with a cosmic gravitational radiation background corresponding to the aforementioned power-law models. More recently [132], a joint analysis of data from ESA’s Planck satellite and the ground-based BICEP2 and Keck Array experiments did not confirm such a finding.

4.3.2. The Effect of Rotating Masses

The previously mentioned solution $\bar{h}_{\mu\nu}$ of the inhomogeneous wave equation in terms of the retarded potentials was used by Thirring [56,57,61] to investigate, to a certain extent, the relative nature of the centrifugal and Coriolis fictitious forces arising in a rotating coordinate system with respect to another one connected with the static background of the fixed stars. Indeed, according to a fully relativistic point of view, they should also be viewed as gravitational effects caused by the rotation of the distant stars with respect to a fixed coordinate system. At first sight, it may seem that such a possibility is already included in the theory itself in view of the covariance of the field equations. Actually, it is not so because the boundary conditions at infinite distance play an essential role in selecting, *de facto*, some

privileged coordinate systems, in spite of a truly “relativistic” spirit with which the theory should be informed. In other words, although the equations of the theory are covariant, the choice of the boundary conditions at spatial infinity, which are distinct from and independent of the field equations themselves, would pick up certain coordinate systems with respect to others, which is a conceptual weakness of an alleged “generally relativistic” theory. Thus, Thirring [56] did not aim to check the full equivalence of the gravitational effects of the rotation of the whole of the distant stars of the Universe with those due to the rotation of the coordinate system with respect to them, assumed fixed. Indeed, he considered just a rotating hollow shell of finite radius D and mass M , so to circumvent the issue of the boundary conditions at infinite distance by setting the spacetime metric tensor there equal to the Minkowskian one. By assuming M small with respect to the whole of the fixed stars, so to consider the departures of the $g_{\mu\nu}$ coefficients from their STR values $\eta_{\mu\nu}$ small inside the shell, the application of the previously obtained Einsteinian expression for $\bar{h}_{\mu\nu}$ to the shell yielded that a test particle inside the hollow space inside it is affected by accelerations which are formally identical to the centrifugal and Coriolis ones, apart from a multiplicative scaling dimensionless factor as little as

$$\frac{GM}{c^2 D} \quad (43)$$

This explains the failures by Newton [133] in attributing the centrifugal curvature of the free surface of water in his swirling bucket to the relative rotation of the bucket itself and the water, and the Friedländer brothers [134] who unsuccessfully attempted to detect centrifugal forces inside a heavy rotating flywheel.

Another application of the approximate solution $\bar{h}_{\mu\nu}$ of the inhomogeneous wave equation allowed to discover that, while in either GTR and the Newtonian theory the gravitational field of a static, spherical body is identical to that of a point mass [81], it is not so in GTR if the body rotates. Indeed, Einstein [53], Thirring and Lense [54] calculated the (tiny) precessions affecting the orbits of test particles as natural satellites and planets moving in the field of rotating astronomical bodies such as the Sun and some of its planets. Such a peculiarity of the motion about mass-energy currents, universally known as “Lense-Thirring effect” by historical tradition (cfr. [62] for a critical historical analysis of its genesis), was subjected to deep experimental scrutiny in the last decades [65–67].

In the sixties of the twentieth century, another consequence of the rotation of an astronomical body was calculated within GTR: the precession of an orbiting gyroscope [135,136], sometimes dubbed as “Pugh-Schiff effect”. The GP-B experiment [137], aimed to directly measure also such an effect in the field of the Earth, was successfully completed a few years ago [64], although the final accuracy obtained ($\sim 19\%$) was worse than that expected ($\sim 1\%$ or better).

4.4. Black Holes and Other Physically Relevant Exact Solutions of the Field Equations

4.4.1. The Reissner-Nordström Metric

In 1916, Reissner [138] solved the coupled Einstein-Maxwell field equations and found the metric which describes the geometry of the spacetime surrounding a pointlike electric charge Q . One year later, Weyl [79] obtained the same metric from a variational action principle. In 1918, Nordström [139], generalized it to the case of a spherically symmetric charged body. The metric for a non-rotating charge

distribution is nowadays known as the Reissner-Nordström metric; in the limit $Q \rightarrow 0$, it reduces to the Schwarzschild solution.

The physical relevance of the Reissner-Nordström metric in astronomical and astrophysical scenarios depends on the existence of macroscopic bodies stably endowed with net electric charges.

4.4.2. Black Holes

One of the consequences of the vacuum Schwarzschild solution was that it predicts the existence of a surface of infinite red-shift at

$$r = r_g \doteq \frac{2GM}{c^2} \quad (44)$$

Thus, if, for some reasons, a body could shrink so much to reduce to such a size, it would disappear from the direct view of distant observers, who would not be anymore able to receive any electromagnetic radiation from such a surface, later interpreted as a spatial section of an “event horizon” [140–142]. A “frozen star”, a name common among Soviet scientists from 1958 to 1968 [143,144], would have, then, formed, at least from the point of view of an external observer. In 1968 [143,145], Wheeler renamed such objects with their nowadays familiar appellation of “black holes” [146].

In fact, both Eddington in 1926 [147] and Einstein in 1939 [148], although with arguments at different levels of soundness, were firmly convinced that such bizarre objects could not form in the real world. Instead, in 1939 [149], Oppenheimer and Snyder demonstrated that, when all the thermonuclear sources of energy are exhausted, a sufficiently heavy star will unstoppably collapse beyond its Schwarzschild radius to end in a spacetime singularity. The latter one is not to be confused with the so-called “Schwarzschild singularity” occurring in the Schwarzschild metric at $r = r_g$, which was proven in 1924 [150] to be unphysical, being a mere coordinate artifact; nonetheless, it took until 1933 for Lemaître to realize it [151]. In 1965, Penrose [152], in his first black hole singularity theorem, demonstrated that the formation of a singularity at the end of a gravitational collapse was an inevitable result, and not just some special feature of spherical symmetry. A black hole is the 4-dimensional spacetime region which represents the future of an imploding star: it insists on the 2-dimensional spatial critical surface determined by the star’s Schwarzschild radius. The 3-dimensional spacetime hypersurface delimiting the black hole, *i.e.*, its event horizon, is located in correspondence of the critical surface [142].

4.4.3. The Kerr Metric

In 1963, the third physically relevant *exact* vacuum solution of the Einstein field equations was found by Kerr [153]. It describes the spacetime metric outside a rotating source endowed with mass M and proper angular momentum J . It was later put in a very convenient form by Boyer and Lindquist [154]. At that time, it was generally accepted that a spherical star would collapse to a black hole described by the Schwarzschild metric. Nonetheless, people was wondering if such a dramatic fate of a star undergoing gravitational collapse was merely an artifact of the assumed perfect spherical symmetry. Perhaps, the slightest angular momentum would halt the collapse before the formation of an event horizon, or at least before the formation of a singularity. In this respect, finding a metric for a rotating star would have been quite valuable.

Contrary to the Schwarzschild solution [80], the Kerr one has not yet been satisfactorily extended to the interior of any realistic matter-energy distribution, despite several attempts over the years [155]. Notably, according to some researchers [156–158], this limit may have no real physical consequences since the exterior spacetime of a rotating physically likely source is *not* described by the Kerr metric whose higher multipoles, according to the so-called “no-hair” conjecture [159,160], can all be expressed in terms of M and J [161,162], which is not the case for a generic rotating star [163]. Moreover, the Kerr solution does not represent the metric during any realistic gravitational collapse; rather, it yields the asymptotic metric at late times as whatever dynamical process produced the black hole settles down, contrary to the case of a non-rotating collapsing star whose exterior metric is described by the Schwarzschild metric at all times. The Birkhoff’s Theorem [81] does not hold for the Kerr metric.

The enormous impact that the discovery by Kerr has had in the subsequent fifty years on every subfield of GTR and astrophysics as well is examined in [158]; just as an example, it should be recalled that, at the time of the Kerr’s discovery, the gravitational collapse to a Schwarzschild black hole had difficulty in explaining the impressive energy output of quasars, discovered and characterized just in those years [164,165], because of the “frozen star” behavior for distant observers. Instead, the properties of the event horizon were different with rotation taken into account. A comparison of the peculiar features of the Schwarzschild and the Kerr solutions can be found in [166].

4.4.4. The Kerr-Newman Metric

In 1965 [167], a new *exact* vacuum solution of the Einstein-Maxwell equations of GTR appeared: the Kerr-Newman metric [168]. It was obtained from the Reissner-Nordström metric by a complex transformation algorithm [169] without integrating the field equations, and is both the spinning generalization of Reissner-Nordström and the electrically charged version of the Kerr metric. Such solutions point towards the possibility that charged and rotating bodies can undergo gravitational collapse to form black holes just as in the uncharged, static case of the Schwarzschild metric.

Leaving the issue of its physical relevance for astrophysics applications out of consideration, the Kerr-Newman metric is the most general static/stationary black hole solution to the Einstein-Maxwell equations. Thus, it is of great importance for theoretical considerations within the mathematical framework of GTR and beyond. Furthermore, understanding this solution also provides valuable insights into the other black hole solutions, in particular the Kerr metric.

5. Application to Cosmology

5.1. Difficulties of Newtonian Cosmologies

The birth of modern cosmology might be dated back to the correspondence between Newton and Bentley in the last decade of the seventeenth century [170], when the issue of the applicability of Newtonian gravitational theory to a static, spatially infinite (Euclidean) Universe uniformly filled with matter was tackled. In four letters to R. Bentley, Newton explored the possibility that matter might be spread uniformly throughout an infinite space. To the Bentley’s suggestion that such an even distribution might be stable, Newton replied that, actually, matter would tend to collapse into large massive bodies.

However, he apparently also thought that they could be stably spread throughout all the space. In particular, in his letter of 10 December 1692, Newton wrote [171]: “it seems to me that if [...] all the matter in the Universe was evenly scattered throughout all the heavens, & every particle had an innate gravity towards all the rest & the whole space throughout which this matter was scattered, was but finite: the matter on the outside of this space would by its gravity tend towards all the matter on the inside & by consequence fall down to the middle of the whole space & there compose one great spherical mass But if the matter was evenly diffused through an infinite space, it would never convene into one mass but some of it convene into one mass & some into another so as to make an infinite number of great masses scattered at great distances from one another throughout all that infinite space.”

Connected with the possibility that matter would fill uniformly an infinite space, and, thus, indirectly with the application of Newtonian gravitation to cosmology, there was also the so-called Olbers paradox [172], some aspects of which had been previously studied also by Kepler [173], Halley [174,175] and de Chéseaux [176]. According to it, although the light from stars diminishes as the square of the distance to the star, the number of stars in spherical shells increases as the square of the shell's radius. As a result, the accumulated effect of the light intensity should make the night sky as bright as the surface of the Sun. In passing, the Olbers paradox touched also other topics which will become crucial in contemporary cosmology like the temporal infinity of the Universe and its material content, and its spatial infinity as well. At the end of the nineteenth century, Seeliger [177] showed that, in the framework of the standard Newtonian theory, matter cannot be distributed uniformly throughout an infinite Universe. Instead, its density should go to zero at spatial infinity faster than r^{-2} ; otherwise, the force exerted on a point mass by all the other bodies of the Universe would be undeterminate because it would be given by a non-convergent, oscillating series. Later, Einstein [178] critically remarked that, if the potential was finite at large distances as envisaged by Seeliger [177] to save the Newtonian law, statistical considerations would imply a depopulation of the fixed stars ensemble, assumed initially in statistical equilibrium. The possibility of an infinite potential at large distances, corresponding to a finite or vanishing not sufficiently fast matter density, already ruled out by Seeliger himself, was excluded also by Einstein [178] because it would yield unrealistically fast speeds of the distant stars. Seeliger [179] demonstrated also that matter density could be different from zero at arbitrary distances if the standard Poisson equation was modified as

$$\nabla^2 \Phi - \Lambda \Phi = 4\pi G\rho \quad (45)$$

It admits

$$\Phi = -\frac{4\pi G\rho}{\Lambda} \quad (46)$$

as a viable solution for a uniform matter density, thus making an evenly filled Universe stable. For a discussion of the problems encountered by the Newtonian theory of gravitation to cosmology, see, e.g., [180].

The inadequacy of Newtonian gravitation to cosmological problems can be also inferred in view of the modern discoveries concerning the expansion of the Universe over the eons (see Section 5.2) which, in conjunction with the finite value of c , yielded to the notion of observable Universe. As previously recalled in Section 3.2, the gravitational interaction among macroscopic bodies can be adequately described, to the first approximation, by the non-relativistic Newtonian model. Such an approximation is applicable over spatial scales ranging from laboratory to planetary, stellar, and galactic systems. On

the other hand [181], the Newtonian model cannot be applied, not even to the first approximation, to correctly describe gravity over cosmological distances of the order the Hubble distance

$$D_H \doteq \frac{c}{H_0} \sim 10^{26} \text{ m} \quad (47)$$

where [182]

$$H_0 = (67.3 \pm 1.2) \text{ km s}^{-1} \text{ Mpc}^{-1} \quad (48)$$

is the current value of the Hubble parameter (see Section 5.4), which fixes the maximum spatial distance accessible to current observations (the radius of the observable universe is proportional to D_H through a numerical coefficient which, according to the present-day cosmological parameters, is equal to 3.53). Indeed, the absolute value of the potential of the mass equivalent to the energy density ε enclosed in a spherical volume of radius $\sim D_H$ is

$$|\Phi_H| = \frac{4}{3} \frac{\pi G \varepsilon}{H_0^2} \quad (49)$$

The condition of validity of the Newtonian approximation is that, for any test particle of mass m , the gravitational potential energy $m|\Phi_H|$ resulting from the interaction with the cosmological mass of the observable Universe is much smaller than its rest energy mc^2 . Instead, it turns out [181]

$$\frac{4}{3} \frac{\pi G \varepsilon}{H_0^2 c^2} \sim 1 \quad (50)$$

It follows that the Newtonian approximation is not valid at the Hubble scale, and a correct dynamical description of the Universe to cosmological scales must necessarily rely upon a relativistic theory of gravity.

5.2. Relativistic Cosmological Models

GTR, applied to cosmology for the first time in 1917 by Einstein himself [178], was able to put such a fundamental branch of our knowledge on the firm grounds of empirical science.

In the following, we will try to follow the following terminological stipulations [183]. We will generally use the word “Universe” to denote a model of the cosmological spacetime along with its overall matter-energy content; as we will see, the relativistic Universe is the space woven by time and weighed by all forms of energy (matter-either baryonic and non-baryonic-, radiation, cosmological constant). As such, the Universe has neither center nor borders, neither inside nor outside. Instead, by means of “universe” we will denote the observable portion of the cosmological spacetime delimited by a cosmological horizon unavoidably set by the fact that all the physical means (electromagnetic and gravitational radiation, neutrinos, cosmic rays) by which we collect information from objects around us travel at finite speeds. Its spatial section is a centered on the Earth-based observer with a radius equal to

$$3.53D_H = 51.3 \text{ Gly} = 15.7 \text{ Gpc} \quad (51)$$

5.2.1. The Static Einstein Model

In 1917, Einstein [178] showed that, following his field equations in their original form, it would not be possible to choose the boundary conditions in such a way to overcome simultaneously the

depopulation and the observed small stellar velocities issues. Instead, in principle, it is mathematically possible to modify them in as much as the same way as it was doable with the Poisson equation by introducing a Λ term which yielded

$$R_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \quad (52)$$

Some years later, Cartan [184] demonstrated that the most general form of the Einstein field equations necessarily implies the Λ term. It turned out that a Universe uniformly filled with constant matter density ρ and non-vanishing

$$\Lambda_E = \frac{4\pi G\rho}{c^2} \quad (53)$$

would rest in equilibrium. Moreover, since it would be spatially closed with

$$\begin{aligned} g_{00} &= 1 \\ g_{0i} &= 0 \\ g_{ij} &= - \left[\delta_{ij} + \frac{x_i x_j}{S^2 - (x_1^2 + x_2^2 + x_3^2)} \right] \end{aligned} \quad (54)$$

and radius S connected with Λ by

$$\Lambda = \frac{1}{S^2} \quad (55)$$

there would not be the need of choosing suitable boundary conditions at infinity, thus removing the aforementioned “non-relativistic” drawback of the theory (see Section 4). It should be noted that if such a 4-dimensional cylindrical Universe did not contain matter, there would not be any gravitational field, *i.e.*,

$$T_{\mu\nu} = 0 \quad (56)$$

would imply

$$g_{\mu\nu} = 0 \quad (57)$$

Thus, the postulate of the complete relativity of inertia would be met. In the Einstein spatially hyperspherical model, the spacetime trajectories of moving bodies and light rays wind around spirals on the surface of a cylinder in such a way that if one watched a spaceship moving away from her/him, it first would diminish in size but then would come back beginning to magnify again. Thirteen years later, Eddington [185] showed that the static Einsteinian model is, actually, unstable.

It may be interesting to note [20] how the Einstein’s Universe is, in fact, no less liable to the Olbers paradox than the Newtonian one; indeed, the light emitted by a star would endlessly circumnavigate the static spherical space until obstructed by another star.

5.2.2. The de Sitter Model

In 1917, de Sitter [186,187] found a solution for the modified Einstein field equations with $\Lambda \neq 0$ yielding a 4-dimensional hyperbolic Universe

$$g_{\mu\nu} = \frac{\eta_{\mu\nu}}{\left(1 - \frac{\Lambda}{12} \eta_{\alpha\beta} x^\alpha x^\beta\right)^2} \quad (58)$$

$$\Lambda = \frac{3}{S^2}$$

with non-zero gravitational field even in absence of matter, thus differing from the Einstein model. It allowed also a sort of spatial (and not material) origin of inertia, which would be relative to void space: a hypothetical single test particle existing in the otherwise empty de Sitter Universe would have inertia just because of Λ .

At the time of the Einstein and de Sitter models, there were not yet *compelling* means to observationally discriminate between them [188], although their physical consequences were remarkably different. Suffice it to say that the spacetime geometry of the de Sitter Universe implied that, although static, test particles would have escaped far away because of the presence of the Λ term, unless they were located at the origin. Such a recessional behaviour was known as “de Sitter effect”. As said by Eddington [189], “the de Sitter Universe contains motion without matter, while the Einstein Universe contains matter without motion”.

After having lost appeal with the advent of the genuine non-stationary Fridman-Lemaître solutions (see Section 5.2.3), the de Sitter model was somewhat revamped in the framework of the inflationary phase characterized by an ultrafast expansion that it is believed to have occurred in the early stages of the universe [190–192].

5.2.3. The Fridman-Lemaître-Robertson-Walker Expanding Models

In the twenties of the last century, the first truly non-static theoretical models of the Universe were proposed by Fridman [193,194]. Indeed, he found new solutions of the Einstein field equations with Λ representing spatially homogeneous and isotropic cosmological spacetimes filled with matter-energy modeled as a perfect fluid generally characterized only by time-varying density $\rho(t)$, and endowed with an explicitly time-dependent universal scaling factor $S(t)$ for the spatial metric having constant curvature $k = 0, \pm 1$ throughout all space. If $k = +1$, the 3-dimensional space is *spherical* and necessarily *finite* (as the hypersphere); if $k = 0$, it is *Euclidean*; if $k = -1$, it is *hyperbolic*. Euclidean and hyperbolic spaces can be either finite or infinite, depending on their topology [195,196] which, actually, is *not* determined by the Einstein field equations governing only the dynamical evolution of $\rho(t)$, $S(t)$. Importantly, viable solutions exist also in absence of the cosmological Λ term for all the three admissible values of the spatial curvature parameter k . The Einstein and de Sitter models turned out [193] to be merely limiting cases of an infinite family of solutions of the Einstein field equations for a positive, time-varying matter density, any one of which would imply, at least for a certain time span, a general recession-or oncoming, since the solutions are symmetric with respect to time reversal-of test particle. According to their dynamical behaviour, the Fridman’s models are classified as *closed* if they recollapse, *critical* if they expand at an asymptotically zero rate, and *open* if they expand indefinitely. In this respect, a spherical universe can be open if Λ is positive and large enough, but it cannot be infinite. Conversely, Euclidean or hyperbolic universes, generally open, can be closed if $\Lambda < 0$; their finiteness or infiniteness depends on their topology, not on their material content. Fridman’s simplifying assumptions were much weaker than those of either Einstein and de Sitter, so that they defined a much likelier idealization of the real world [20], as it turned out years later: indeed, the russian scientist was interested only in the mathematical aspects of the cosmological solutions of the Einstein equations.

Approximately in the same years, a body of observational evidence pointing towards mutual recessions of an increasingly growing number of extragalactic nebulae was steadily accumulating [197–201] from accurate red-shifts measurements, probably unbeknownst to Fridman. In 1929, Hubble [200] made his momentous claim that the line-of-sight speeds of the receding galaxies are proportional to their distances from the Earth. If, at first, the de Sitter model, notwithstanding its material emptiness, was regarded with more favor than the Einstein one as a possible explanation of the observed red-shifts of distant nebulae, despite the cautiousness by de Sitter himself [187], it would have been certainly superseded by the more realistic Fridman ones, if only they had been widely known at that time (Fridman died in 1925). It may be that a role was played in that by the negative remark by Einstein about a claimed incompatibility of the non-stationary Fridman's models with his field equations [202], later retracted by the father of GTR because of an own mathematical error in his criticism [203].

At any rate, in 1927, Lemaître [204], who apparently did never hear of the Fridman's solutions, rederived them and applied them to the physical universe with the explicit aim of founding a viable explanation of the observed recessions of galaxies (the red-shifted nebulae had been recognized as extra-galactic objects analogous to our own galaxy in 1925 by Hubble [205]). Lemaître [204] also showed that the static solution by Einstein is unstable with respect to a temporal variation of matter density. Enlightened by the Hubble's discovery [200], and, perhaps, also struck by the criticisms by Lemaître [204] and Eddington [185] to his own static model, Einstein fully acknowledged the merits of the non-static Fridman-type solutions rejecting outright his cosmological Λ term as unnecessary and unjustifiable [206].

Interestingly, in 1931, Lemaître [207] did not appreciate the disown by Einstein of his cosmological constant Λ , which, instead, was retained by the belgian cosmologist as an essential ingredient of the physical Universe for a number of reasons, one of which connected also with quantum mechanics, which, however, convinced neither Einstein nor the scientific community, at least until the end of the nineties of the last century [208]. His “hesitating” model was characterized by positive spatial curvature ($k = +1$), and a positive cosmological constant so that its perpetual expansion is first decelerated, then it enters an almost stationary state, and finally it accelerates, thus resolving the problem of the age of the Universe and the time required for the formation of galaxies.

The formal aspects of the homogeneous and isotropic expanding models were clarified and treated in a systematic, general approach in the first half of the thirties of the last century by Robertson [209–211] and Walker [212]. Today, the spacetime tensor g of standard expanding cosmologies is commonly named as Fridman-Lemaître-Robertson-Walker (FLRW) metric (see Section 5.4).

5.3. The Einstein-de Sitter Model

In 1932, Einstein and de Sitter [213] published a brief note of two pages whose aim was to simplify the study of cosmology. About their work, as reported by Eddington [214], Einstein would have told him: “I did not think the paper very important myself, but de Sitter was keen on it”, while de Sitter wrote to him: “You will have seen the paper by Einstein and myself. I do not myself consider the result of much importance, but Einstein seemed to think that it was”. At any rate, such an exceedingly simplified solution, characterized by dust-like, pressureless matter, $k = 0$, $\Lambda = 0$ and perpetual, decelerating

expansion, served as “standard model” over about six decades, to the point of curb researches on other models. In it, mutual distances among test particles grow as $t^{2/3}$. Such a behaviour is unstable in the sense that it can only occur if $k = 0$ exactly; for tiny deviations from such a value, the expansion would gradually depart from the trajectory of the Einstein-de Sitter model. Actually, it represented the best description of the cosmic expansion as it was known for the next sixty years. The fact that the observed behaviour of the physical universe was still so close to that particular expansion rate suggested that the instability had not yet had the time to manifest itself significantly. But, after all, the universe had been expanding for about several billions of years, as if it had started just from the very spacial initial conditions of the Einstein-de Sitter model. This peculiar situation, later known as “the flatness problem”, motivated, among other things, the studies on the cosmic inflation in the eighties of the last century [190–192]. The Einstein-de Sitter model has now been abandoned, also because it would imply a too short age of the Universe given by

$$t_0 = \frac{2}{3} \frac{1}{H_0} = 9.6 \text{ Gyr} \quad (59)$$

For a recent popular account on the panoply of possible Universes allowed by GTR, see [215].

In passing, let us note that the expanding cosmological models by GTR, along with the associated finite age of the Universe, represent the framework to correctly solve the Olbers paradox [216,217].

5.4. Some Peculiar Characteristics of the FLRW Models

The assumptions of homogeneity and isotropy of the spatial sections of the FLRW models are of crucial importance. It must be stressed that they are, in general, distinct requirements. Homogeneity does not generally imply isotropy: for instance, think about a universe filled with galaxies whose axes of rotation are all aligned along some specific spatial direction, or a wheat field where the ears grow all in the same direction. Conversely, a space which is isotropic around a certain point, in the sense that the curvature is the same along all the directions departing from it, may well not be isotropic in other points, or, if some other points of isotropy exist, the curvature there can be different from each other: an ovoid surface is not homogenous since its curvature varies from point to point, but the space is isotropic around its two “vertices”. Instead, the same value of the curvature in all the directions, *i.e.*, the same amount of isotropy, around all points of space implies homogeneity [218]. As far as our location is concerned, it can be said phenomenologically that isotropy about us holds in several physical aspects to a high level of accuracy, as demonstrated, e.g., by the CMB which is isotropic at a 10^{-5} level. In view of the Copernican spirit, it is commonly postulated that every other observer located everywhere would see the same situation, thus assuring the homogeneity as well: it is the content of the so-called Cosmological Principle. The fact that the curvature of the spatial parts of the FLRW models is the same everywhere, and that they are expanding over time, according to the Weyl principle [219], admit a peculiar foliation of the spacetime which allows for an unambiguous identification of the spatial sections of simultaneity and of the bundle of time-like worldlines orthogonal to them as worldlines of fundamental observers at rest marking a common, cosmic time. Thus, it is possible to describe the spacetime of the Universe as

the mathematical product of a 3-dimensional Riemannian space with the temporal axis. In comoving dimensionless spatial coordinates r , θ , ϕ , the line element can be written as

$$(ds)^2 = c^2(dt)^2 - S(t) \left[\frac{(dr)^2}{1 - kr^2} + r^2(d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2 \right] \quad (60)$$

The Einstein field equations, applied to the FLRW metric with a pressureless cosmic fluid as standard source with matter and radiation densities ρ_m , ρ_r , respectively, yield the Fridman equation

$$\dot{S}^2 = \frac{8}{3}\pi G (\rho_m + \rho_r) S^2 - kc^2 + \frac{1}{3}\Lambda c^2 S^2 \quad (61)$$

By defining the Hubble parameter as

$$H \doteq \frac{\dot{S}}{S} \quad (62)$$

and the critical density as

$$\rho_{\text{crit}} \doteq \frac{3H^2}{8\pi G} \quad (63)$$

it is possible to recast the Fridman equation in the form

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1 \quad (64)$$

or also

$$\Omega_{\text{tot}} = 1 - \Omega_k \quad (65)$$

by posing

$$\Omega_{\text{tot}} \doteq \Omega_m + \Omega_r + \Omega_\Lambda \quad (66)$$

with the dimensionless parameters entering Equation (66) defined as

$$\begin{aligned} \Omega_m &\doteq \frac{8}{3} \frac{\pi G \rho_m}{H^2} > 0 \\ \Omega_r &\doteq \frac{8}{3} \frac{\pi G \rho_r}{H^2} > 0 \\ \Omega_\Lambda &\doteq \frac{1}{3} \frac{\Lambda c^2}{H^2} \leq 0 \\ \Omega_k &\doteq -\frac{kc^2}{S^2 H^2} \leq 0 \end{aligned} \quad (67)$$

At present epoch, $\Omega_{r,0} \sim 0$, so that the normalized Fridman equation reduces to

$$\Omega_{m,0} + \Omega_{\Lambda,0} + \Omega_{k,0} = 1 \quad (68)$$

or also

$$\Omega_{\text{tot},0} = 1 - \Omega_{k,0} \quad (69)$$

Results collected in the last twenty years from a variety of observational techniques (e.g., SNe Ia [220–222], Baryon acoustic oscillations [223], WMAP [224], Planck [182]), interpreted within a FRLW framework, point towards an observable universe whose spatial geometry is compatible with an Euclidean one (such a possibility, in view of the unavoidable error bars, is impossible to be proved experimentally with certainty: on the contrary, it could be well excluded should the ranges of values

for $\Omega_{\text{tot},0}$ did not contain 1), and whose dynamical behaviour is characterized by a small positive cosmological constant Λ which makes it accelerating at late times. By assuming $\Omega_{\text{tot},0} = 1$ *exactly*, as allowed by the experimental data and predicted by the inflationary paradigm, the values for the other normalized densities are inferred by finding [182,224]

$$\Omega_{\text{m},0} \sim 0.3, \Omega_{\Lambda,0} \sim 0.7 \quad (70)$$

6. Summary

After its birth, GTR went to fertilize and seed, directly as well as indirectly, many branches of disparate sciences as mathematics [225–230], metrology [231–234], geodesy [236–238], geophysics [239–241], astronomy [242–247], astrophysics [248–252], cosmology [181,253–255], not to say about the exquisite technological spin-off [256–270] due to the long-lasting efforts required to put to the test various key predictions of the theory [18,271]. Moreover, once some of them have been or will be successfully tested, they have or will become precious tools for determine various parameters characterizing several natural systems, often in extreme regimes inaccessible with other means: gravitational microlensing for finding extrasolar planets, even of terrestrial size [272,273], weak and strong gravitational lensing to map otherwise undetectable matter distributions over galactic, extragalactic and cosmological scales [274–276], frame-dragging to measure angular momenta of spinning objects like stars and planets [277–280], gravitational waves to probe, e.g., quantum gravity effects [281], modified models of gravity [282,283] and cosmic inflationary scenarios [131,132], to characterize tight binary systems hosting compact astrophysical objects like white dwarves, neutron stars and black holes [284–289], and to investigate extremely energetic events like, e.g., supernovæ explosions [290].

However, GTR has its own limits of validity, and presents open problems [291]. At certain regimes, singularities plague it [225,292–295]. Connected to this issue, there is also a major drawback of the theory of gravitation of Einstein, *i.e.*, its lingering inability to merge with quantum mechanics yielding a consistent theory of quantum gravity [296–302]. Moreover, in view of the discoveries made in the second half of the last century about the seemingly missing matter to explain the rotation curves of galaxies [303–305] and the accelerated expansion of the Universe [220–222], it might be that GTR need to be modified [306–311] also at astrophysical and cosmological scales in order to cope with the issue of the so-called “dark” [312] components of the matter-energy content of the Universe known as Dark Matter and Dark Energy.

We consider it appropriate to stop here with our sketchy review. Now, we give the word to the distinguished researchers who will want to contribute to this Special Issue by bringing us towards the latest developments of the admirable and far-reaching theory of gravitation by Einstein. At a different level of coverage and completeness, the interested reader may also want to consult the recent two-volume book [313,314].

References

1. Einstein, A. Zur allgemeinen Relativitätstheorie. *Sitzungsber. Kön. Preuß. Akad. Wiss. zu Berlin* **1915**, 778–786. (In German)

2. Einstein, A. Zur allgemeinen Relativitätstheorie (Nachtrag). *Sitzungsber. Kön. Preuß. Akad. Wiss. zu Berlin* **1915**, 799–801. (In German)
3. Einstein, A. Die Feldgleichungen der Gravitation. *Sitzungsber. Kön. Preuß. Akad. Wiss. zu Berlin* **1915**, 844–847. (In German)
4. Damour, T. General Relativity Today. In *Gravitation and Experiment. Poincaré Seminar 2006. Progress in Mathematical Physics. Volume 52*; Damour, T., Duplantier, B., Rivasseau, V., Eds.; Birkhäuser: Basel, Switzerland, 2007; pp. 1–49.
5. Torretti, R. The Geometric Structure of the Universe. In *Philosophy and the Origin and Evolution of the Universe*; Agazzi, E., Cordero, A., Eds.; Kluwer Academic Publisher: Dordrecht, The Netherlands, 1991; pp. 53–73.
6. Lorentz, H.A. Electromagnetische verschijnselen in een stelsel dat zich met willekeurige snelheid, kleiner dan die van het licht, beweegt. *Koninklijke Akademie van Wetenschappen te Amsterdam. Wis-en Natuurkundige Afdeeling. Verslagen van de Gewone Vergaderingen* **1904**, *12*, 986–1009. (In Dutch)
7. Poincaré, H. Sur la dynamique de l' électron. *Comptes-rendus des séances de l'Académie des Sci.* **1905**, *140*, 1504–1508. (In French)
8. Poincaré, H. Sur la dynamique de l' électron. *Rendiconti del Circolo Matematico di Palermo* **1906**, *21*, 129–175. (In French)
9. Einstein, A. Zur Elektrodynamik bewegter Körper. *Ann. Phys.* **1905**, *17*, 891–921. (In German)
10. Preti, G.; de Felice, F.; Masiero, L. On the Galilean non-invariance of classical electromagnetism. *Eur. J. Phys.* **2009**, *30*, 381–391.
11. Planck, M. Die Kaufmannschen Messungen der Ablenkbarkeit der β -Strahlen in ihrer Bedeutung für die Dynamik der Elektronen. *Phys. Z.* **1906**, *7*, 753–761. (In German)
12. Bucherer, A.H. Ein Versuch, den Elektromagnetismus auf Grund der Relativbewegung darzustellen. *Phys. Z.* **1906**, *7*, 553–557. (In German)
13. Stachel, J.; Cassidy, D.C.; Renn, J.; Schulmann, R. Editorial Note: Einstein on the Theory of Relativity. In *The Collected Papers of Albert Einstein. Volume 2: The Swiss Years: Writings, 1900–1909*; Princeton University Press: Princeton, NJ, USA, 1989; pp. 253–274.
14. Holton, G.; Elkana, Y. *Albert Einstein: Historical and Cultural Perspectives-The Centennial Symposium in Jerusalem*; Princeton University Press: Princeton, NJ, USA, 1982; p. xv.
15. Klein, F. Über der geometrischen Grundlagen der Lorentzgruppe. *Jahresbericht der Deutschen Mathematiker-Vereinigung* **1910**, *19*, 281–300. (In German)
16. Fock, V.A. *Theory of Space, Time, and Gravitation*, 2nd ed.; Pergamon Press: New York, NY, USA, 1964.
17. Cartan, E. La théorie des groupes et la géometrie. *L'enseignement Math.* **1927**, *26*, 200–225. (In French)
18. Will, C.M. The Confrontation between General Relativity and Experiment. *Living Rev. Relativ.* **2006**, *163*, 146–162.
19. Pauli, W. Relativitätstheorie. In *Encyklopädie der mathematischen Wissenschaften*; Teubner: Leipzig, Germany, 1921; volume 5, pt. 2, art. 19. (In German)
20. Torretti, R. *Relativity and Geometry*; Dover: Mineola, NY, USA, 1996; pp. 33–34.

21. Renn, J. *Auf den Schultern von Riesen und Zwergen. Einsteins unvollendete Revolution*; Wiley-VCH: Weinheim, Germany, 2006. (In German)
22. Newton, IS. *Philosophiæ Naturalis Principia Mathematica*; For the Royal Society by Joseph Streater: London, UK, 1687. (In Latin)
23. Poisson, S.D. Remarques sur une équation qui se présente dans la théorie des attractions des sphéroïdes. *Nouveau Bull. Soc. Philomath. Paris* **1813**, *3*, 388–392. (In French)
24. Newton, Is. *Original letter from Isaac Newton to Richard Bentley. 25 February 1692/3*; Trinity College Cambridge 189.R.4.47, ff. 7-8: Trinity College Library: Cambridge, UK, 1692. Available online: <http://www.newtonproject.sussex.ac.uk/view/texts/normalized/THEM00258> (accessed on 20 April 2015).
25. Maxwell, J.C. *A Treatise on Electricity and Magnetism*; Clarendon Press: Oxford, UK, 1873.
26. Einstein, A. Autobiographische Skizze. In *Helle Zeit-Dunkle Zeit. In Memoriam Albert Einstein*; Seelig, C., Ed.; Europa Verlag: Zürich, Switzerland, 1956; pp. 9–17. (In German)
27. Abraham, M. Una nuova teoria della gravitazione. *Il Nuovo Cimento*. **1912**, *4*, 459–481. (In Italian)
28. Einstein, A. Über den Einfluß der Schwerkraft auf die Ausbreitung des Lichtes. *Ann. Phys.* **1911** *340*, 898–908. (In German)
29. Einstein, A. Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen. *Jahrbuch der Radioaktivität* **1907**, *4*, 411–462. (In German)
30. von Eötvös, R. Über die Anziehung der Erde auf verschiedene Substanzen. *Mathematische und Naturwissenschaftliche Berichte aus Ungarn* **1890**, *8*, 65–68. (In German)
31. Adler, C.G.; Coulter, B.L. Galileo and the Tower of Pisa experiment. *Am. J. Phys.* **1978**, *46*, 199–201.
32. Segre, M. Galileo, Viviani and the tower of Pisa. *Stud. History Philos. Sci. Part A*. **1989**, *20*, 435–451.
33. Crease, R.P. The Legend of the Leaning Tower. In *The Laws of Motion. An Anthology of Current Thought*; Hall, L.E., Ed.; The Rosen Publishing Group: New York, NY, USA, 2006; pp. 10–14.
34. Drake, S. *Galileo at Work. His Scientific Biography*; University of Chicago Press: Chicago, IL, USA, 1978.
35. Chandrasekhar, S. *Newton's Principia for the Common Reader*; Clarendon Press: Oxford, UK, 1995; p. 362.
36. Chandrasekhar, S. *Newton's Principia for the Common Reader*; Clarendon Press: Oxford, UK, 1995; p. 364.
37. Peeters, C. Gothic *kaurus*, Sanskrit *guruḥ*, Greek *βαρύς*. *Indogermanische Forschungen* **2010**, *79*, 33–34.
38. Chandrasekhar, S. *Newton's Principia for the Common Reader*; Clarendon Press: Oxford, UK, 1995; p. 369.
39. Mie, G. Grundlagen einer Theorie der Materie. *Ann. Phys.* **1913**, *40*, 1–66. (In German)
40. Pound, R.V.; Rebka, G.A. Apparent Weight of Photons. *Phys. Rev. Lett.* **1960**, *4*, 337–341.
41. Corda, C. Interpretation of Mössbauer experiment in a rotating system: A new proof for general relativity. *Ann. Phys.-New York* **2015**, *355*, 360–366.

42. Blamont, J.E.; Roddier, F. Precise Observation of the Profile of the Fraunhofer Strontium Resonance Line. Evidence for the Gravitational Red Shift on the Sun. *Phys. Rev. Lett.* **1961**, *7*, 437–439.
43. Adams, W.S. The Relativity Displacement of the Spectral Lines in the Companion of Sirius. *Proc. Natl. Acad. Sci. USA* **1925**, *11*, 382–387.
44. Einstein, A. Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie. *Sitzungsber. Kön. Preuß. Akad. Wiss. zu Berlin* **1915**, 831–839. (In German)
45. Dyson, F.W.; Eddington, A.S.; Davidson, C. A Determination of the Deflection of Light by the Sun's Gravitational Field, from Observations Made at the Total Eclipse of May 29, 1919. *Philos. Trans. R. Soc. London. Series A*. **1920**, *220*, 291–333.
46. Will, C.M. The 1919 measurement of the deflection of light. *Class. Quantum Gravit.* **2015**, in press.
47. Klein, M.J.; Kox, A.J.; Renn, J.; Schulman, R. Appendix A. Scratch Notebook, 1909–1914?. In *The Collected Papers of Albert Einstein. Volume 3: The Swiss Years: Writings 1909–1911*; Princeton University Press: Princeton, NJ, USA, 1994; p. 585.
48. Renn, J.; Sauer, T.; Stachel, J. The Origin of Gravitational Lensing: A Postscript to Einstein's 1936 Science paper. *Science* **1997**, *275*, 184–186.
49. Walsh, D.; Carswell, R.F.; Weymann, R.J. 0957 + 561 A, B: Twin quasistellar objects or gravitational lens?. *Nature* **1979**, *279*, 381–384.
50. Einstein, A. Gibt es eine Gravitationswirkung, die der elektrodynamischen Induktionswirkung analog ist? *Vierteljahrschrift für gerichtliche Medizin und öffentliches Sanitätswesen* **1912**, *44*, 37–40. (In German)
51. Einstein, A. Zum gegenwärtigen Stande des Gravitationsproblem. *Phys. Z.* **1913**, *14*, 1249–1266. (In German)
52. Klein, M.J.; Kox, A.J.; Schulmann, R. Einstein and Besso: Manuscript on the Motion of the Perihelion of Mercury. In *The Collected Papers of Albert Einstein. Volume 4. The Swiss Years: Writings, 1912–1914*; Princeton University Press: Princeton, NJ, USA, 1995; pp. 360–473.
53. Schulmann, R.; Kox, A.J.; Janssen, M.; Illy, J. To Hans Thirring. In *The Collected Papers of Albert Einstein. Volume 8. The Berlin Years: Correspondence, 1914–1918*; Princeton University Press: Princeton, NJ, USA, 1998; pp. 500–501 and pp. 564–566.
54. Lense, J.; Thirring, H. Über den Einfluß der Eigenrotation der Zentralkörper auf die Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie. *Phys. Z.* **1918**, *19*, 156–163. (In German)
55. Thirring, H. Über die formale Analogie zwischen den elektromagnetischen Grundgleichungen und den Einsteinschen Gravitationsgleichungen erster Näherung, *Phys. Z.* **1918**, *19*, 204–205. (In German).
56. Thirring, H. Über die Wirkung rotierender ferner Massen in der Einsteinschen Gravitationstheorie. *Phys. Z.* **1918**, *19*, 33–39. (In German)
57. Thirring, H. Berichtigung zu meiner Arbeit: “Über die Wirkung rotierender ferner Massen in der Einsteinschen Gravitationstheorie”. *Phys. Z.* **1921**, *22*, 29–30. (In German)

58. Mashhoon, B. On the gravitational analogue of Larmor's theorem. *Phys. Lett. A* **1993**, *173*, 347–354.
59. Iorio, L. A gravitomagnetic effect on the orbit of a test body due to the earth's variable angular momentum. *Int. J. Mod. Phys. D* **2002**, *11*, 781–787.
60. Bini, D.; Cherubini, C.; Chicone, C.; Mashhoon, B. Gravitational induction. *Class. Quantum Gravit.* **2008**, *25*, 225014.
61. Mashhoon, B.; Hehl, F.W.; Theiss, D.S. On the Influence of the Proper Rotation of Central Bodies on the Motions of Planets and Moons According to Einstein's Theory of Gravitation. *Gen. Relativ. Gravit.* **1984**, *16*, 727–741.
62. Pfister, H. On the history of the so-called Lense-Thirring effect. *Gen. Relativ. Gravit.* **2007**, *39*, 1735–1748.
63. Pfister, H. Editorial note to: Hans Thirring, On the formal analogy between the basic electromagnetic equations and Einstein's gravity equations in first approximation. *Gen. Relativ. Gravit.* **2012**, *44*, 3217–3224.
64. Everitt, C.W.F.; Debra, D.B.; Parkinson, B.W.; Turneare, J.P.; Conklin, J.W.; Heifetz, M.I.; Keiser, G.M.; Silbergleit, A.S.; Holmes, T.; Kolodziejczak, J.; *et al.* Gravity Probe B: Final Results of a Space Experiment to Test General Relativity. *Phys. Rev. Lett.* **2011**, *106*, 221101.
65. Ciufolini, I. Dragging of inertial frames. *Nature* **2007**, *449*, 41–47.
66. Iorio, L.; Lichtenegger, H.I.M.; Ruggiero, M.L.; Corda, C. Phenomenology of the Lense-Thirring effect in the solar system. *Astrophys. Space Sci.* **2011**, *331*, 351–395.
67. Renzetti, G. History of the attempts to measure orbital frame-dragging with artificial satellites. *Cent. Eur. J. Phys.* **2013**, *11*, 531–544.
68. Rindler, W. The Lense-Thirring effect exposed as anti-Machian. *Phys. Lett. A* **1994**, *187*, 236–238.
69. Bondi, H.; Samuel, J. The Lense-Thirring effect and Mach's principle. *Phys. Lett. A* **1997**, *228*, 121–126.
70. Rindler, W. The case against space dragging. *Phys. Lett. A* **1997**, *233*, 25–29.
71. Lichtenegger, H.I.M.; Mashhoon, B. Mach's Principle. In *The Measurement of Gravitomagnetism: A Challenging Enterprise*; Iorio, L., Ed.; Nova: Hauppauge, NY, USA, 2007; pp. 13–25.
72. Einstein, A.; Grossmann, M. Kovarianzeigenschaften der Feldgleichungen der auf die verallgemeinerte Relativitätstheorie gegründeten Gravitationstheorie. *Z. Math. Phys.* **1914**, *63*, 215–225. (In German)
73. Landau, L.D.; Lifšitis, E.M. *Teoriya Polia*; Nauka: Moscow, Russia, 1973. (In Russian)
74. Hilbert, D. Die Grundlagen der Physik (Erste Mitteilung). *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physikalische Klasse* **1915**, 395–407. (In German)
75. Corry, L.; Renn, J.; Stachel, J. Belated Decision in the Hilbert-Einstein Priority Dispute. *Science* **1997**, *278*, 1270–1273.
76. Le Verrier, U. Lettre de M. Le Verrier à M. Faye sur la théorie de Mercure et sur le mouvement du périhélie de cette planète. *Cr. Hebd. Acad. Sci.* **1859**, *49*, 379–383. (In French)

77. Schwarzschild, K. Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungsber. Kön. Preuß. Akad. Wiss. zu Berlin* **1916**, 189–196 (In German)
78. Droste, J. Het veld van een enkel centrum in Einstein's theorie der zwaartekracht, en de beweging van een stoffelijk punt in dat veld. *Koninklijke Akademie van Wetenschappen te Amsterdam. Wis-en Natuurkundige Afdeeling. Verslagen van de Gewone Vergaderingen* **1916**, 25, 163–180. (In Dutch)
79. Weyl, H. Zur Gravitationstheorie. *Ann. Phys.* **1917**, 359, 117–145. (In German)
80. Schwarzschild, K. Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie. *Sitzungsber. Kön. Preuß. Akad. Wiss. zu Berlin* **1916**, 424–434. (In German)
81. Birkhoff, G.D. *Relativity and Modern Physics*; Harvard University Press: Cambridge, MA, USA, 1923.
82. Brush, S.G. Prediction and Theory Evaluation: The Case of Light Bending. *Science* **1989**, 246, 1124–1129.
83. Weinberg, S. *Dreams of a final theory*; Pantheon Books: New York, NY, USA, 1992; pp. 96–97.
84. Einstein, A. Die Grundlage der allgemeinen Relativitätstheorie. *Ann. Phys.* **1916**, 354, 769–822. (In German)
85. De Sitter, W. On Einstein's Theory of Gravitation and its Astronomical Consequences. Second Paper. *Mon. Not. R. Astron. Soc.* **1916**, 77, 155–184.
86. Schouten, J.A. Over het ontstaan eener praecessiebeweging tengevolge van het niet euklidisch zijn der ruimte in de nabijheid van de zon. *Koninklijke Akademie van Wetenschappen te Amsterdam. Wis-en Natuurkundige Afdeeling. Verslagen van de Gewone Vergaderingen* **1918**, 27, 214–220. (In Dutch)
87. Fokker, A.D. De geodetische precessie: Een uitvloeisel van Einstein's gravitatiethorie. *Koninklijke Akademie van Wetenschappen te Amsterdam. Wis-en Natuurkundige Afdeeling. Verslagen van de Gewone Vergaderingen* **1920**, 29, 611–621. (In Dutch)
88. Bertotti, B.; Ciufolini, I.; Bender, P.L. New test of general relativity-Measurement of de Sitter geodetic precession rate for lunar perigee. *Phys. Rev. Lett.* **1987**, 58, 1062–1065.
89. Williams, J.G.; Newhall, X.X.; Dickey, J.O. Relativity parameters determined from lunar laser ranging. *Phys. Rev. D* **1996**, 53, 6730–6739.
90. Shapiro, I.I. Fourth Test of General Relativity. *Phys. Rev. Lett.* **1964**, 13, 789–791.
91. Shapiro, I.I.; Pettengill, G.H.; Ash, M.E.; Stone, M.L.; Smith, W.B.; Ingalls, R.P.; Brockelman, R.A. Fourth Test of General Relativity: Preliminary Results. *Phys. Rev. Lett.* **1968**, 20, 1265–1269.
92. Bertotti, B.; Iess, L.; Tortora, P. A test of general relativity using radio links with the Cassini spacecraft. *Nature* **2003**, 425, 374–376.
93. Einstein, A. Näherungsweise Integration der Feldgleichungen der Gravitation. *Sitzungsber. Kön. Preuß. Akad. Wiss. zu Berlin* **1916**, 688–696. (In German)
94. Will, C.M. Propagation Speed of Gravity and the Relativistic Time Delay. *Astrophys. J.* **2003**, 590, 683–690.

95. Carlip, S. Model-dependence of Shapiro time delay and the “speed of gravity/speed of light” controversy. *Class. Quantum Gravit.* **2005**, *21*, 3803–3812.
96. Kopeikin, S.M. Comment on ‘Model-dependence of Shapiro time delay and the “speed of gravity/speed of light” controversy’. *Class. Quantum Gravit.* **2005**, *22*, 5181–5186.
97. Kopeikin, S.M. Testing the Relativistic Effect of the Propagation of Gravity by Very Long Baseline Interferometry. *Astrophys. J.* **2001**, *556*, L1–L5.
98. Fomalont, E.B.; Kopeikin, S.M. The Measurement of the Light Deflection from Jupiter: Experimental Results. *Astrophys. J.* **2003**, *598*, 704–711.
99. Einstein, A. Über Gravitationswellen. *Sitzungsber. Kön. Preuß. Akad. Wiss. zu Berlin* **1918**, 154–167 (In German)
100. Taylor, J.H.; Fowler, L.A.; McCulloch, P.M. Measurements of general relativistic effects in the binary pulsar PSR 1913+16. *Nature* **1979**, *277*, 437–440.
101. Taylor, J.H.; Weisberg, J.M. A new test of general relativity-Gravitational radiation and the binary pulsar PSR 1913+16. *Astrophys. J.* **1982**, *253*, 908–920.
102. Weisberg, J.M.; Taylor, J.H. Observations of Post-Newtonian Timing Effects in the Binary Pulsar PSR 1913+16. *Phys. Rev. Lett.* **1984**, *52*, 1348–1350.
103. Hulse, R.A.; Taylor, J.H. Discovery of a pulsar in a binary system. *Astrophys. J.* **1975**, *195*, L51–L53.
104. Damour, T. 1974: The discovery of the first binary pulsar. *Class. Quantum Gravit.* **2015**, In press.
105. Pitkin, M.; Reid, S.; Rowan, S.; Hough, J. Gravitational Wave Detection by Interferometry (Ground and Space). *Living Rev. Relativ.* **2011**, *14*, 5.
106. Freise, A.; Strain, K.A. Interferometer Techniques for Gravitational-Wave Detection. *Living Rev. Relativ.* **2010**, *13*, 1.
107. Losurdo, G. Ground-based gravitational wave interferometric detectors of the first and second generation: An overview. *Class. Quantum Gravit.* **2012**, *29*, 124005.
108. Acernese, F.; *et al.* Advanced Virgo: A second-generation interferometric gravitational wave detector. *Class. Quantum Gravit.* **2015**, *31*, 024001.
109. Abbott, B.P.; Abbott, R.; Adhikari, R.; Ajith, P.; Allen, B.; Allen, G.; Amin, R.S.; Anderson, S.B.; Anderson, W.G.; Arain, M.A.; *et al.* LIGO: The Laser Interferometer Gravitational-Wave Observatory. *Rep. Progr. Phys.* **2009**, *72*, 076901.
110. Hammond, G.; Hild, S.; Pitkin, M. Advanced technologies for future ground-based, laser-interferometric gravitational wave detectors. *J. Mod. Optics* **2014**, *61*, S10–S45.
111. Gair, J.R.; Vallisneri, M.; Larson, S.L.; Baker, J.G. Testing General Relativity with Low-Frequency, Space-Based Gravitational-Wave Detectors. *Liv. Rev. Relativ.* **2013**, *16*, 7.
112. Vitale, S. Space-borne gravitational wave observatories. *Gen. Relativ. Gravit.* **2014**, *46*, 1730.
113. Armstrong, J.W. Low-Frequency Gravitational Wave Searches Using Spacecraft Doppler Tracking. *Living Rev. Relativ.* **2006**, *9*, 1.
114. Porter, E.K. The Challenges in Gravitational Wave Astronomy for Space-Based Detectors. In *Gravitational Wave Astrophysics*; Sopuerta, C., Ed.; Springer: Berlin, Germany, 2015; pp. 267–279.

115. Amaro-Seoane, P.; Aoudia, S.; Babak, S.; Binétruy, P.; Berti, E.; Bohé, A.; Caprini, C.; Colpi, M.; Cornish, N.J.; Danzmann, K.; *et al.* eLISA: Astrophysics and cosmology in the millihertz regime. *GW Notes* **2013**, *6*, 4–110.
116. Ni, W.-T. Astrod-Gw Overview and Progress. *Int. J. Mod. Phys. D* **2013**, *22*, 1341004.
117. Loeb, A.; Maoz, D. Using Atomic Clocks to Detect Gravitational Waves. **2015**, arXiv:1501.00996.
118. Thorne, K.S. Sources of Gravitational Waves and Prospects for their Detection. In *Recent Advances in General Relativity*; Janis, A.I., Porter, J.R., Eds.; Birkhäuser: Boston, MA, USA, 1992; pp. 196–229.
119. Gertsenshtein, M.E.; Pustovoit, V.I. K voprosu ob obnaruzhenii gravitatsionnykh valn malykh chastot. *Zh. Eksp. Teor. Fiz.* **1962**, *43*, 605–607. (In Russian)
120. Weber, J. Evidence for Discovery of Gravitational Radiation. *Phys. Rev. Lett.* **1969**, *22*, 1320–1324.
121. Weber, J. Gravitational-Wave-Detector Events. *Phys. Rev. Lett.* **1968**, *20*, 1307–1308.
122. Damour, T. An Introduction to the Theory of Gravitational Radiation. In *Gravitation in Astrophysics*; Carter, B., Hartle, J.B., Eds.; Plenum Press: New York, NY, USA, 1987; pp. 3–62.
123. Corda, C. Interferometric Detection of Gravitational Waves: the Definitive Test for General Relativity. *Int. J. Mod. Phys. D* **2009**, *18*, 2275–2282.
124. Sathyaprakash, B.S.; Schutz, B.F. Physics, Astrophysics and Cosmology with Gravitational Waves. *Living Rev. Relativ.* **2009**, *12*, 2.
125. Press, W.H.; Thorne, K.S. Gravitational-Wave Astronomy. *Ann. Rev. Astron. Astrophys.* **1972**, *10*, 335–374.
126. Gasperini, M. *Relatività Generale e Teoria della Gravitazione*, 2nd ed.; Springer: Milan, Italy, 2015. (In Italian)
127. Penzias, A.A.; Wilson, R.W. A Measurement of Excess Antenna Temperature at 4080 Mc/s. *Astrophys. J.* **1965**, *142*, 419–421.
128. Dodelson, S. *Modern Cosmology*; Academic Press: San Diego, CA, USA, 2003.
129. Kamionkowski, M.; Kosowsky, A.; Stebbins, A. Statistics of cosmic microwave background polarization. *Phys. Rev. D* **1997**, *55*, 7368–7388.
130. Seljak, U.; Zaldarriaga, M. Signature of Gravity Waves in the Polarization of the Microwave Background. *Phys. Rev. Lett.* **1997**, *78*, 2054–2057.
131. BICEP2 Collaboration. Detection of B-Mode Polarization at Degree Angular Scales by BICEP2. *Phys. Rev. Lett.* **2014**, *112*, 241101.
132. BICEP2/Keck, Planck Collaborations. A Joint Analysis of BICEP2/Keck Array and Planck Data. *Phys. Rev. Lett.* **2015**, In press.
133. Newton, I. *Scholium to the Definitions in Philosophiae Naturalis Principia Mathematica, Bk. 1 (1689)*; trans. Andrew Motte (1729), rev. Florian Cajori; University of California Press: Berkeley, CA, USA, 1934; pp. 6–12. Available online: <http://plato.stanford.edu/entries/newton-stm/scholium.html> (accessed on 20 April 2015).
134. Friedländer, B.; Friedländer, I. *Absolute Oder Relative Bewegung?*; Simion: Berlin, Germany, 1896. (In German)

135. Pugh, G.E. Proposal for a Satellite Test of the Coriolis Prediction of General Relativity. *Weapons Systems Evaluation Group. Research Memorandum No. 11*; The Pentagon: Washington, DC, USA, 1959.
136. Schiff, L.I. Possible New Experimental Test of General Relativity Theory. *Phys. Rev. Lett.* **1960**, *4*, 215–217.
137. Everitt, C.W.F. The Gyroscope Experiment. I: General Description and Analysis of Gyroscope Performance. In *Proceedings of the International School of Physics “Enrico Fermi”. Course LVI. Experimental Gravitation*; Bertotti, B., Ed.; Academic Press: New York, NY, USA, 1974; pp. 331–402.
138. Reissner, H. Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie. *Ann. Phys.* **1916**, *355*, 106–120. (In German)
139. Nordström, G. Een an Ander Over de Energie van het Zwaartekrachtsveld Volgens de Theorie van Einstein. *Koninklijke Akademie van Wetenschappen te Amsterdam. Wis-en Natuurkundige Afdeling. Verslagen van de Gewone Vergaderingen* **1918**, *26*, 1201–1208. (In Dutch)
140. Rindler, W. Visual Horizons in World Models. *Mon. Not. R. Astron. Soc.* **1956**, *116*, 662–677.
141. Finkelstein, D. Past-Future Asymmetry of the Gravitational Field of a Point Particle. *Phys. Rev.* **1958**, *110*, 965–967.
142. Brill, D. Black Hole Horizons and How They Begin. *Astron. Rev.* **2012**, *7*, 1–12.
143. Wheeler, J.A.; Gearhart, M. FORUM: From the Big Bang to the Big Crunch. *Cosmic Search* **1979**, *1*, 2–8.
144. Thorne, K.S. *From Black Holes to Time Warps: Einstein’s Outrageous Legacy*; W.W. Norton & Company: New York, NY, USA, 1994; pp. 255–256.
145. Wheeler, J.A. Our universe: The known and the unknown. *Am. Sci.* **1968**, *56*, 1–20.
146. Ruffini, R.J.; Wheeler, J.A. Introducing the black hole. *Phys. Today* **1971**, *24*, 30–41.
147. Eddington, A.S. *The Internal Constitution of the Stars*; Cambridge University Press: Cambridge, UK, 1926; p. 6.
148. Einstein, A. On a Stationary System with Spherical Symmetry Consisting of Many Gravitating Masses. *Ann. Math.* **1939**, *40*, 922–936.
149. Oppenheimer, J.R.; Snyder, H. On Continued Gravitational Contraction. *Phys. Rev.* **1939**, *56*, 455–459.
150. Eddington, A.S. A Comparison of Whitehead’s and Einstein’s Formulæ. *Nature* **1924**, *113*, 192.
151. ’t Hooft, G. *Introduction to the Theory of Black Holes*; Spinoza Institute: Utrecht, The Netherlands, 2009; pp. 47–48.
152. Penrose, R. Gravitational collapse and space-time singularities. *Phys. Rev. Lett.* **1965**, *14*, 57–59.
153. Kerr, R.P. Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics. *Phys. Rev. Lett.* **1963**, *11*, 237–238.
154. Boyer, R.H.; Lindquist, R.W. Maximal Analytic Extension of the Kerr Metric. *J. Math. Phys.* **1967**, *8*, 265–281.
155. Krasiński, A. Ellipsoidal Space-Times, Sources for the Kerr Metric. *Ann. Phys.-New York* **1978**, *112*, 22–40.

156. Thorne, K.S. Relativistic Stars, Black Holes and Gravitational Waves (Including an In-Depth Review of the Theory of Rotating, Relativistic Stars). In *Proceedings of the International School of Physics “Enrico Fermi”. Course XLVII. General Relativity and Cosmology*; Sachs, B.K., Ed.; Academic Press: New York, NY, USA, 1971; pp. 237–283.
157. Boshkayev, K.; Quevedo, H.; Ruffini, R. Gravitational field of compact objects in general relativity. *Phys. Rev. D* **2012**, *86*, 064043.
158. Teukolsky, S.A. The Kerr Metric. *Class. Quantum Gravit.* **2015**, in press.
159. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*; W.H. Freeman: San Francisco, CA, USA, 1973; pp. 875–876.
160. Heusler, M. Uniqueness Theorems for Black Hole Space-Times. In *Black Holes: Theory and Observation*; Hehl, F.W., Kiefer, C., Metzler, R.J.K., Eds.; Springer: Berlin, Germany, 1998; pp. 157–186.
161. Geroch, R. Multipole Moments. II. Curved Space. *J. Math. Phys.* **1970**, *11*, 2580–2588.
162. Hansen, R.O. Multipole moments of stationary space-times. *J. Math. Phys.* **1974**, *15*, 46–52.
163. Hartle, J.B.; Thorne, K.S. Slowly Rotating Relativistic Stars. II. Models for Neutron Stars and Supermassive Stars. *Astrophys. J.* **1968**, *153*, 807–834.
164. Matthews, T.A.; Sandage, A.R. Optical Identification of 3C 48, 3C 196, and 3C 286 with Stellar Objects. *Astrophys. J.* **1963**, *138*, 30–58.
165. Chiu, H.-Y. Gravitational Collapse. *Phys. Today* **1964**, *17*, 21–34.
166. Heinicke, C.; Hehl, F.W. Schwarzschild and Kerr solutions of Einstein’s field equation: An Introduction. *Int. J. Mod. Phys. D* **2015**, *24*, 1530006.
167. Newman, E.T.; Couch, R.; Chinnapared, K.; Exton, A.; Prakash, A.; Torrence, R. Metric of a Rotating, Charged Mass. *J. Math. Phys.* **1965**, *6*, 918–919.
168. Newman, E.T.; Adamo, T. Kerr-Newman metric. *Scholarpedia* **2014**, *9*, 31791.
169. Newman, E.T.; Janis, A.I. Note on the Kerr spinning particle metric. *J. Math. Phys.* **1965**, *6*, 915–917.
170. Ryan, M.P.; Shepley, L.C. *Homogeneous Relativistic Cosmologies*; Princeton University Press: Princeton, NJ, USA, 1975.
171. Newton, Is. *Original letter from Isaac Newton to Richard Bentley, dated 10 December 1692*; Trinity College Cambridge 189.R.4.47, ff. 4A-5; Trinity College Library: Cambridge, UK, 1692. Available online: <http://www.newtonproject.sussex.ac.uk/view/texts/normalized/THEM00254> (accessed on 20 April 2015).
172. Olbers, H.W. Ueber die Durchsichtigkeit des Weltraums, von Hrn. Dr. Olbers in Bremen, unterm 7. Mai 1823. eingesandt. In *Astronomisches Jahrbuch für das Jahr 1826*; Bode, J.E., Ed.; Ferd. Dümmler: Berlin, Germany, 1823; pp. 110–121. (In German)
173. IOANNIS KEPLERI Mathematici Cæsarei. *DISSERTATIO Cum NVNCIO SIDEREO nuper ad mortales misso à GALILÆO GALILÆO Mathematico Patauino*; Sedesanus: Prague, Czech Republic, 1610. (In Latin)
174. Halley, E. On the Infinity of the Sphere of the Fix’d Stars. *Philos. Trans.* **1720** *31*, 22–24.
175. Halley, E. Of the Number, Order and Light of the Fix’d Stars. *Philos. Trans.* **1720** *31*, 24–26.

176. De Chéseaux, J.P.L. *Traité de la comète: qui a paru en décembre 1743 & en janvier, février & mars 1744 / contenant outre les observations de l'auteur, celles qui ont été faites à Paris par Mr. Cassini, & à Genève par Mr. Calandrini; on y a joint diverses observations & dissertations astronomiques; le tout accompagné de figures en taille douce*; Marc-Michel Bousquet & compagnie: Lausanne, Switzerland, 1744; Appendix 2. (In French)
177. Von Seeliger, H. Ueber das Newton'sche Gravitationsgesetz. *Astron. Nachr.* **1895**, *137*, 129–136. (In German)
178. Einstein, A. Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie. *Sitzungsber. Kön. Preuß. Akad. Wiss. zu Berlin* **1917**, 142–152. (In German)
179. Von Seeliger, H. Über das Newton'sche Gravitationsgesetz. *Sitzungsberichte der Königlich Bayerische Akademie der Wissenschaften zu München. Mathematisch-physikalische Klasse* **1896**, *126*, 373–400. (In German)
180. Norton, J.D. The Cosmological Woes of Newtonian Gravitation Theory. In *The Expanding Worlds of General Relativity*; Goenner, H., Renn, J., Ritter, T., Sauer, T., Eds.; Birkhäuser: Basel, Switzerland, 1999; pp. 271–322.
181. Gasperini, M. *Lezioni di Cosmologia Teorica*; Springer: Milan, Italy, 2012. (In Italian)
182. Planck Collaboration. Planck 2013 results. XVI. Cosmological parameters. *Astron. Astrophys.* **2014**, *571*, A16.
183. Luminet, J.-P. *L'Unives chiffonné*; Gallimard: Paris, France, 2005. (In French)
184. Cartan, E. Sur les équations de la gravitation d'Einstein. *J. Math. Pures Appl.* **1922**, *9*, 141–203.
185. Eddington, A.S. On the Instability of Einstein's Spherical World. *Mon. Not. R. Astron. Soc.* **1930**, *90*, 668–678.
186. de Sitter, W. Over de relativiteit der traagheid: Beschouwingen naar aanleiding van Einstein's laatste hypothese. *Koninklijke Akademie van Wetenschappen te Amsterdam. Wis-en Natuurkundige Afdeeling. Verslagen van de Gewone Vergaderingen* **1917**, *25*, 1268–1276. (In Dutch)
187. de Sitter, W. Over de kromming der ruimte. *Koninklijke Akademie van Wetenschappen te Amsterdam. Wis-en Natuurkundige Afdeeling. Verslagen van de Gewone Vergaderingen* **1917**, *26*, 222–236. (In Dutch)
188. de Sitter, W. Over de mogelijkheid van statistisch evenwicht van het heelal. *Koninklijke Akademie van Wetenschappen te Amsterdam. Wis-en Natuurkundige Afdeeling. Verslagen van de Gewone Vergaderingen* **1920**, *29*, 651–653. (In Dutch)
189. Eddington, A.S. *The Expanding Universe*; The University of Michigan Press: Ann Arbor, MI, USA, 1958, p. 46.
190. Starobinsky, A.A. Spektr reliktovogo gravitacionnogo izlucheniya i nachal'noe sostoyanie Vselennoi. *Zh. Eksp. Teor. Fiz.* **1979**, *30*, 719–723. (In Russian)
191. Guth, A.H. Inflationary universe: A possible solution to the horizon and flatness problems. *Phys. Rev. D* **1981**, *23*, 347–356.
192. Linde, A.D. A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems. *Phys. Lett. B* **1982**, *108*, 389–393.
193. Friedman, A. Über die Krümmung des Raumes. *Zeitschr. Phys.* **1922**, *10*, 377–386. (In German)

194. Friedman, A. Über die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes. *Zeitschr. Phys.* **1924**, *21*, 326–332. (In German)
195. Lachieze-Rey, M.; Luminet, J.-P. Cosmic topology. *Phys. Rep.* **1995**, *254*, 135–214.
196. Luminet, J.-P. Cosmic topology: Twenty years after. *Grav. Cosmol.* **2014**, *20*, 15–20.
197. Slipher, V.M. On the spectrum of the nebula in the Pleiades. *Lowell Obs. Bull.* **1912**, *2*, 26–27.
198. Slipher, V.M. The radial velocity of the Andromeda Nebula. *Lowell Obs. Bull.* **1913**, *1*, 56–57.
199. Slipher, V.M. Nebulae. *Proc. Am. Philos. Soc.* **1917**, *56*, 403–409.
200. Hubble, E. A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae. *Proc. Natl. Acad. Sci. USA* **1929**, *15*, 168–173.
201. Humason, M.L. Apparent Velocity-Shifts in the Spectra of Faint Nebulae. *Astrophys. J.* **1931**, *74*, 35–42.
202. Einstein, A. Bemerkung zu der Arbeit von A. Friedmann “Über die Krümmung des Raumes”. *Zeitschr. Phys.* **1922**, *11*, 326. (In German)
203. Einstein, A. Notiz zu der Arbeit von A. Friedmann “Über die Krümmung des Raumes”. *Zeitschr. Phys.* **1923**, *16*, 228. (In German)
204. Lemaître, G. Un Univers homogène de masse constante et de rayon croissant rendant compte de la vitesse radiale des nébuleuses extra-galactiques. *Annales de la Société Scientifique de Bruxelles* **1927**, *47*, 49–59. (In French)
205. Hubble, E. NGC 6822, a remote stellar system. *Astrophys. J.* **1925**, *62*, 409–433.
206. Einstein, A. Zum kosmologischen Problem der allgemeinen Relativitätstheorie. *Sitzungsber. Kön. Preuß. Akad. Wiss. zu Berlin* **1931**, 235–237. (In German)
207. Lemaître, G. L’ étrangeté de l’Univers. **1960**, *La Revue Generale Belge*, 1–14. (In French)
208. Luminet, J.-P. *L’invention du Big Bang*; Editions du Seuil: Paris, France, 2004. (in French)
209. Robertson, H.P. On the Foundations of Relativistic Cosmology. *Proc. Natl. Acad. Sci. USA* **1929**, *15*, 822–829.
210. Robertson, H.P. Relativistic Cosmology. *Rev. Mod. Phys.* **1933**, *5*, 62–90.
211. Robertson, H.P. Kinematics and World-Structure. *Astrophys. J.* **1935**, *82*, 284–301.
212. Walker, A.G. On the formal comparison of Milne’s kinematical system with the systems of general relativity. *Mon. Not. R. Astron. Soc.* **1935**, *95*, 263–269 .
213. Einstein, A.; de Sitter, W. On the Relation between the Expansion and the Mean Density of the Universe. *Proc. Natl. Acad. Sci. USA* **1932**, *18*, 213–214.
214. Eddington, A.S. Forty Years of Astronomy. In *Background to Modern Science*; Needham, J., Page, W., Eds.; Cambridge University Press: Cambridge, UK, 1938; p. 128.
215. Barrow, J.D. *The Book of Universes*; The Bodley Head: London, UK, 2011.
216. Wesson, P.S. The real reason the night sky is dark: Correcting a myth in astronomy teaching. *J. Br. Astron. Ass.* **1989**, *99*, 10–13.
217. Harrison, E. *Cosmology. The Science of the Universe*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2000; pp. 491–514.
218. Weinberg, S. *Cosmology*; Oxford University Press: Oxford, UK, 2008.
219. Weyl, H. Zur allgemeinen Relativitätstheorie. *Zeitschr. Phys.* **1923**, *24*, 230–232. (In German)

220. Riess, A.G.; Filippenko, A.V.; Challis, P.; Clocchiatti, A.; Diercks, A.; Garnavich, P.M.; Gilliland, R.L.; Hogan, C.J.; Jha, S.; Kirshner, R.P.; *et al.* Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant. *Astron. J.* **1998**, *116*, 1009–1038.

221. Schmidt, B.P.; Suntzeff, N.B.; Phillips, M.M.; Schommer, R.A.; Clocchiatti, A.; Kirshner, R.P.; Garnavich, P.; Challis, P.; Leibundgut, B.; Spyromilio, J.; *et al.* The High-Z Supernova Search: Measuring Cosmic Deceleration and Global Curvature of the Universe Using Type IA Supernovae. *Astrophys. J.* **1998**, *507*, 46–63.
222. Perlmutter, S.; Aldering, G.; Goldhaber, G.; Knop, R.A.; Nugent, P.; Castro, P.G.; Deustua, S.; Fabbro, S.; Goobar, A.; Groom, D.E.; *et al.* Measurements of Ω and Λ from 42 High-Redshift Supernovae. *Astrophys. J.* **1999**, *517*, 565–586.
223. Padmanabhan, N.; Xu, X.; Eisenstein, D.J.; Scalzo, R.; Cuesta, A.J.; Mehta, K.T.; Kazin, E. A 2 per cent distance to $z = 0.35$ by reconstructing baryon acoustic oscillations - I. Methods and application to the Sloan Digital Sky Survey. *Mon. Not. R. Astron.* **2012**, *427*, 2132–2145.
224. Hinshaw, G.; Larson, D.; Komatsu, E.; Spergel, D.N.; Bennett, C.L.; Dunkley, J.; Nolte, M.R.; Halpern, M.; Hill, R.S.; Odegard, N.; *et al.* Nine-year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results. *Astrophys. J. Suppl. Ser.* **2013**, *208*, 19.
225. Hawking, S.W.; Penrose, R. The Singularities of Gravitational Collapse and Cosmology. *Proc. R. Soc. Lond. A* **1970**, *314*, 529–548.
226. Hawking, S.W.; Ellis, G.F.R. *The Large Scale Structure of Space-time*; Cambridge University Press: Cambridge, UK, 1973.
227. Stephani, A.; Kramer, D.; Mac Callum, M.A.H.; Hoenselaers, C.; Herlt, E. *Exact Solutions of Einstein's Field Equations*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2003.
228. Mac Callum, M.A.H. Exact solutions of Einstein's equations. *Scholarpedia* **2013**, *8*, 8584.
229. Kopeikin, S.; Efroimsky, M.; Kaplan, G. Relativistic Reference Frames. In *Relativistic Celestial Mechanics of the Solar System*; Kopeikin, S., Efroimsky, M., Kaplan, G., Eds.; Wiley-VCH: Berlin, Germany, 2011; pp. 371–428.
230. Kopeikin, S.; Efroimsky, M.; Kaplan, G. Post-Newtonian Coordinate Transformations. In *Relativistic Celestial Mechanics of the Solar System*; Kopeikin, S., Efroimsky, M., Kaplan, G., Eds.; Wiley-VCH: Berlin, Germany, 2011; pp. 249–462.
231. Ashby, N. Relativity in the Global Positioning System. *Living Rev. Relativ.* **2003**, *6*, 1.
232. Guinot, B. Metrology and general relativity. *Comptes Rendus Phys.* **2004**, *5*, 821–828.
233. Larson, K.M.; Ashby, N.; Hackman, C.; Bertiger, W. An assessment of relativistic effects for low Earth orbiters: The GRACE satellites. *Metrologia* **2007**, *44*, 484–490.
234. Reynaud, S.; Salomon, C.; Wolf, P. Testing General Relativity with Atomic Clocks. *Space Sci. Rev.* **2009**, *148*, 233–247.
235. Ahmadi, M.; Bruschi, D.E.; Sabín, C.; Adesso, G.; Fuentes, I. Relativistic Quantum Metrology: Exploiting relativity to improve quantum measurement technologies. *Nat. Sci. Rep.* **2014**, *4*, 4996.
236. Müller, J.; Soffel, M.; Klioner, S.A. Geodesy and relativity. *J. Geodesy* **2008**, *82*, 133–145.
237. Kopeikin, S.; Efroimsky, M.; Kaplan, G. Relativistic Geodesy. In *Relativistic Celestial Mechanics of the Solar System*; Kopeikin, S., Efroimsky, M., Kaplan, G., Eds.; Wiley-VCH: Berlin, Germany, 2011; pp. 671–714.

238. Combrinck, L. General Relativity and Space Geodesy. In *Sciences of Geodesy - II Innovations and Future Developments*; Xu, G., Ed.; Springer: Berlin, Germany, 2013; pp. 53–95.
239. De Sabbata, V.; Fortini, P.; Gualdi, C.; Petralia, S. A Proposal for Combined Efforts Regarding Geophysical Research and Detection of Gravitational Waves. *Ann. Geophys.* **1970**, *23*, 21–25.
240. Grishchuk, L.; Kulagin, V.; Rudenko, V.; Serdobolski, A. Geophysical studies with laser-beam detectors of gravitational waves. *Class. Quantum Grav.* **2005**, *22*, 245–269.
241. Gusev, A.V.; Rudenko, V.N.; Tsybankov, I.V.; Yushkin, V.D. Detection of gravitational geodynamic effects with gravitational-wave interferometers. *Grav. Cosmol.* **2011**, *17*, 76–79.
242. Soffel, M.H. *Relativity in Astrometry, Celestial Mechanics and Geodesy*; Springer: Berlin, Germany, 1989.
243. Brumberg, V.A. *Essential Relativistic Celestial Mechanics*; Adam Hilger: Bristol, UK, 1991.
244. Kopeikin, S. Relativistic astrometry. *Scholarpedia* **2011**, *6*, 11382.
245. Kopeikin, S.; Efroimsky, M.; Kaplan, G. Relativistic Celestial Mechanics. In *Relativistic Celestial Mechanics of the Solar System*; Kopeikin, S., Efroimsky, M., Kaplan, G., Eds.; Wiley-VCH: Berlin, Germany, 2011; pp. 463–518.
246. Kopeikin, S.; Efroimsky, M.; Kaplan, G. Relativistic Astrometry. In *Relativistic Celestial Mechanics of the Solar System*; Kopeikin, S., Efroimsky, M., Kaplan, G., Eds.; Wiley-VCH: Berlin, Germany, 2011; pp. 519–669.
247. Gai, M.; Vecchiato, A.; Ligorì, S.; Sozzetti, A.; Lattanzi, M.G. Gravitation astrometric measurement experiment. *Exp. Astron.* **2012**, *34*, 165–180.
248. Chandrasekhar, S. *The Mathematical Theory of Black Holes*; Oxford University Press: New York, NY, USA, 1983.
249. Harrison, B.K.; Thorne, K.S.; Wakano, M.; Wheeler, J.A. *Gravitation Theory and Gravitational Collapse*; University of Chicago Press: Chicago, IL, USA, 1965.
250. Zeldovich, Ya.B.; Novikov, I.D. *Stars and Relativity*; University of Chicago Press: Chicago, IL, USA, 1971.
251. Colpi, M.; Casella, P.; Gorini, V.; Moschella, U.; Possenti, A. *Physics of Relativistic Objects in Compact Binaries: From Birth to Coalescence*; Springer: Berlin, Germany, 2009.
252. Rezzolla, L.; Zanotti, O. *Relativistic Hydrodynamics*; Oxford University Press: Oxford, UK, 2013.
253. Mukhanov, V. *Physical Foundations of Cosmology*; Cambridge University Press: Cambridge, UK, 2005.
254. Plebański, J.; Kosiński, A. *An Introduction to General Relativity and Cosmology*; Cambridge University Press: Cambridge, UK, 2006.
255. Ellis, G.F.R.; Maartens, R.; Mac Callum, M.A.H. *Relativistic Cosmology*; Cambridge University Press: Cambridge, UK, 2012.
256. Fairbank, W.M. The Use of Low-Temperature Technology in Gravitational Experiments. In *Proceedings of the International School of Physics “Enrico Fermi”. Course LVI. Experimental Gravitation*; Bertotti, B., Ed.; Academic Press: New York, NY, USA, 1974; pp. 280–293.

257. Lipa, A.; Fairbank, W.M.; Everitt, C.W.F. The Gyroscope Experiment - II: Development of the London-Moment Gyroscope and of Cryogenic Technology for Space. In *Proceedings of the International School of Physics “Enrico Fermi”. Course LVI. Experimental Gravitation*; Bertotti, B., Ed.; Academic Press: New York, NY, USA, 1974; pp. 361–380.
258. Stedman, G.E. Ring-laser tests of fundamental physics and geophysics. *Rep. Prog. Phys.* **1997**, *60*, 615–688.
259. Stedman, G.E.; Schreiber, K.U.; Bilger, H.R. On the detectability of the Lense-Thirring field from rotating laboratory masses using ring laser gyroscope interferometers. *Class. Quantum Grav.* **2003**, *20*, 2527–2540.
260. Stedman, G.E.; Hurst, R.B.; Schreiber, K.U. On the potential of large ring lasers. *Opt. Commun.* **2007**, *279*, 124–129.
261. van Zoest, T.; Gaaloul, N.; Singh, Y.; Ahlers, H.; Herr, W.; Seidel, S.T.; Ertmer, W.; Rasel, E.; Eckart, M.; Kajari, E.; *et al.* Bose-Einstein Condensation in Microgravity. *Science* **2010**, *328*, 1540–1543.
262. Belfi, J.; Beverini, N.; Cuccato, D.; Di Virgilio, A.; Maccioni, E.; Ortolan, A.; Santagata, R. Interferometric length metrology for the dimensional control of ultra-stable ring laser gyroscopes. *Class. Quantum Gravit.* **2014**, *31*, 225003.
263. Koenneker, C. Even Einstein Could Not Have Imagined Technology Used to Directly Detect Gravitational Waves. *Scientific American*, March 21, 2014. Available online: <http://www.scientificamerican.com/article/even-einstein-could-not-have-imagined-technology-used-to-directly-detect-gravitational-waves/> (accessed on 16 February 2015).
264. Bassan, M. *Advanced Interferometers and the Search for Gravitational Waves*; Springer: Berlin, Germany, 2014.
265. Di Virgilio, A.; Allegrini, M.; Beghi, A.; Belfi, J.; Beverini, N.; Bosi, F.; Bouhadeh, B.; Calamai, M.; Carelli, G.; Cuccato, D.; *et al.* A ring lasers array for fundamental physics. *Phys. Rev. D* **2014**, *15*, 866–874.
266. Pegna, R.; Nobili, A.M.; Shao, M.; Turyshchev, S.G.; Catastini, G.; Anselmi, A.; Spero, R.; Doravari, S.; Comandi, G.L.; de Michele, A. Abatement of Thermal Noise due to Internal Damping in 2D Oscillators with Rapidly Rotating Test Masses. *Phys. Rev. Lett.* **2011**, *107*, 200801.
267. Islam, M.R.; Ali, M.M.; Lai, M.-H.; Lim, K.-S.; Ahmad, H. Chronology of Fabry-Perot Interferometer Fiber-Optic Sensors and Their Applications: A Review. *Sensors* **2014**, *14*, 7451–7488.
268. Graham, P.W.; Hogan, J.M.; Kasevich, M.A.; Rajendran, S. New Method for Gravitational Wave Detection with Atomic Sensors. *Phys. Rev. Lett.* **2013**, *110*, 171102.
269. Maselli, A.; Gualtieri, L.; Pani, P.; Stella, L.; Ferrari, V. Testing Gravity with Quasi Periodic Oscillations from accreting Black Holes: The Case of Einstein-Dilaton-Gauss-Bonnet Theory. *Astrophys. J.* **2015**. In press.
270. The Extraordinary Technologies of GP-B. Available online: <https://einstein.stanford.edu/TECH/technology1.html> (accessed on 20 April 2015).

271. Everitt, C.W.F.; Huber, M.C.H.; Kallenbach, R.; Schaefer, G.; Schutz, B.F.; Treumann, R.A. *Probing the Nature of Gravity. Confronting Theory and Experiment in Space*; Springer: New York, NY, USA, 2010.
272. Gould, A.; Loeb, A. Discovering planetary systems through gravitational microlenses. *Astrophys. J.* **1992**, *396*, 104–114.
273. Beaulieu, J.-P.; Bennett, D.P.; Fouqué, P.; Williams, A.; Dominik, M.; Jørgensen, U.G.; Kubas, D.; Cassan, A.; Coutures, C.; Greenhill, J.; *et al.* Discovery of a cool planet of 5.5 Earth masses through gravitational microlensing. *Nature* **2006**, *439*, 437–440.
274. Massey, R.; Rhodes, J.; Ellis, R.; Scoville, N.; Leauthaud, A.; Finoguenov, A.; Capak, P.; Bacon, D.; Aussel, H.; Kneib, J.-P.; *et al.* Dark matter maps reveal cosmic scaffolding. *Nature* **2007**, *445*, 286–290.
275. Munshi, D.; Valageas, P.; van Waerbeke, L.; Heavens, A. Cosmology with weak lensing surveys. *Phys. Rep.* **2008**, *462*, 67–121.
276. Hoekstra, H.; Jain, B. Weak Gravitational Lensing and Its Cosmological Applications. *Ann. Rev. Nucl. Part. Syst.* **2008**, *58*, 99–123.
277. Iorio, L. Juno, the angular momentum of Jupiter and the Lense-Thirring effect. *New Astron.* **2010**, *15*, 554–560.
278. Helled, R.; Anderson, J.D.; Schubert, G.; Stevenson, D.J. Jupiter’s moment of inertia: A possible determination by Juno. *Icarus* **2011**, *216*, 440–448.
279. Helled, R. Constraining Saturn’s Core Properties by a Measurement of Its Moment of Inertia-Implications to the Cassini Solstice Mission. *Astrophys. J.* **2011**, *735*, L16.
280. Iorio, L. Constraining the Angular Momentum of the Sun with Planetary Orbital Motions and General Relativity. *Sol. Phys.* **2012**, *281*, 815–826.
281. Alexander, S.; Finn, L.S.; Yunes, N. Gravitational-wave probe of effective quantum gravity. *Phys. Rev. D* **2008**, *78*, 066005.
282. de Laurentis, M.; Capozziello, S.; Nojiri, S.; Odintsov, S.D. PPN limit and cosmological gravitational waves as tools to constrain $f(R)$ -gravity. *Ann. Phys.* **2010**, *19*, 347–350.
283. Damour, T.; Vilenkin, A. Gravitational radiation from cosmic (super)strings: Bursts, stochastic background, and observational windows. *Phys. Rev. D* **2005**, *71*, 063510.
284. Stroeer, A.; Vecchio, A. The LISA verification binaries. *Class. Quantum Gravit.* **2006**, *23*, S809–S817.
285. Pössel, M. Chirping neutron stars. *Einstein Online* **2005**, *1*, 1013. Available online: http://www.einstein-online.info/spotlights/chirping_neutron_stars (accessed on 20 April 2015).
286. Krolak, A. White Dwarf binaries as gravitational wave sources. *Einstein Online* **2010**, *4*, 1001. Available online: <http://www.einstein-online.info/spotlights/galactic-binaries> (accessed on 20 April 2015).
287. Nelemans, G. The Galactic gravitational wave foreground. *Class. Quantum Gravit.* **2009**, *26*, 094030.
288. Sesana, A.; Vecchio, A.; Colacino, C.N. The stochastic gravitational-wave background from massive black hole binary systems: Implications for observations with Pulsar Timing Arrays. *Mon. Not. R. Astron. Soc.* **2008**, *390*, 192–209.

289. Berry, C.P.L.; Gair, J.R. Observing the Galaxy's massive black hole with gravitational wave bursts. *Mon. Not. R. Astron. Soc.* **2012**, *429*, 589–612.
290. Kotake, K.; Sato, K.; Takahashi, K. Explosion mechanism, neutrino burst and gravitational wave in core-collapse supernovae. *Phys. Rep.* **2006**, *69*, 971–1143.
291. Choquet-Bruhat, Y. Results and Open Problems in Mathematical General Relativity. *Milan J. Math.* **2007**, *75*, 273–289.
292. Geroch, R. What is a singularity in general relativity? *Annals Phys.* **1968**, *48*, 526–540.
293. Natario, J. Relativity and Singularities - A Short Introduction for Mathematicians. *Resenhas* **2005**, *6*, 309–335,
294. Uggl, C. Spacetime singularities. *Einstein Online* **2006**, *2*, 1002. Available online: http://www.einstein-online.info/spotlights/singularities/?set_language=en (accessed on 20 April 2015).
295. Curiel, E.; Bokulich, P. Singularities and Black Holes. In *The Stanford Encyclopedia of Philosophy*; Zalta, E.N., Ed.; The Metaphysics Research Lab, Stanford, CA, USA, 2012. Available online: <http://plato.stanford.edu/archives/fall2012/entries/spacetime-singularities/> (accessed on 16 February 2015).
296. Rovelli, C. *Quantum Gravity*; Cambridge University Press: Cambridge, UK, 2004.
297. Ashtekar, A. The winding road to quantum gravity. *Curr. Sci.* **2005**, *89*, 2064–2074.
298. Kiefer, C. Quantum Gravity: General Introduction and Recent Developments. *Ann. Phys.* **2005**, *15*, 129–148.
299. Das, S.; Vagenas, E.C. Universality of Quantum Gravity Corrections. *Phys. Rev. Lett.* **2008**, *101*, 221301.
300. Weinstein, S.; Rickles, D. Quantum Gravity. In *The Stanford Encyclopedia of Philosophy*; Zalta, E.N., Ed.; The Metaphysics Research Lab, Stanford, CA, USA, 2011. Available online: <http://plato.stanford.edu/archives/spr2011/entries/quantum-gravity/> (accessed on 16 February 2015).
301. De Haro, S.; Dieks, D.; 't Hooft, G.; Verlinde, E. Special Issue: Forty Years of String Theory: Reflecting on the Foundations. *Found. Phys.* **2013**, *43*, 1–200.
302. Gambini, R.; Pullin, J. Loop Quantization of the Schwarzschild Black Hole. *Phys. Rev. Lett.* **2013**, *110*, 211301.
303. Bosma, A. 21-cm Line Studies of Spiral Galaxies. II - The Distribution and Kinematics of Neutral Hydrogen in Spiral Galaxies of Various Morphological Types. *Astron. J.* **1981**, *86*, 1825–1846.
304. Rubin, V.C. The Rotation of Spiral Galaxies. *Science* **1983**, *220*, 1339–1344.
305. Salucci, P.; De Laurentis, M. Dark Matter in galaxies: leads to its Nature. *Proceedings of Science* **2012**, *DSU 2012*, 012.
306. Clifton, T.; Barrow, J.D. The power of general relativity. *Phys. Rev. D* **2005**, *72*, 103005.
307. Alimi, J.-M.; Füzfa, A. The abnormally weighting energy hypothesis: The missing link between dark matter and dark energy. *J. Cosmol. Astropart. Phys.* **2008**, *9*, 14.
308. De Felice, A.; Tsujikawa, S. $f(R)$ Theories. *Living Rev. Relativ.* **2010**, *13*, 3.
309. Capozziello, S.; de Laurentis, M. Extended Theories of Gravity. *Phys. Rep.* **2011**, *509*, 167–321.
310. Clifton, T.; Ferreira, P.G.; Padilla, A.; Skordis, C. Modified gravity and cosmology. *Phys. Rep.* **2012**, *513*, 1–189.

- 311. Milgrom, M. The MOND paradigm of modified dynamics. *Scholarpedia* **2014**, *9*, 31410.
- 312. Amendola, L.; Bertone, G.; Profumo, S.; Tait, T. Next Decade in Dark Matter and Dark Energy—Next Decade in Dark Matter and Dark Energy. *Phys. Dark Univ.* **2012**, *1*, 1–218.
- 313. Kopeikin, S.M. *Frontiers in Relativistic Celestial Mechanics. Volume 1 Theory*; De Gruyter: Berlin, Germany, 2014.
- 314. Kopeikin, S.M. *Frontiers in Relativistic Celestial Mechanics. Volume 2 Applications and Experiments*; De Gruyter: Berlin, Germany, 2014.

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